Economics 326
Methods of Empirical Research in Economics

Lecture 5: Gauss-Markov Theorem

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$$\tilde{\beta} = \frac{Y_2 - Y_1}{X_2 - X_1}.$$

- $\tilde{\beta}$ is linear:

$$\tilde{\beta} = c_1 Y_1 + c_2 Y_2,$$

where

$$c_1 = -\frac{1}{X_2 - X_1} \text{ and } c_2 = \frac{1}{X_2 - X_1}.$$

Unbiasedness of $\tilde{\beta}$

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\end{aligned}$$

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&= \beta.
\end{aligned}$$

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- For all $i \neq j$, $E\left(U_i U_j | X_1, \ldots, X_n\right) = 0$.

Then, conditionally on X's, the OLS estimators are BLUE.

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- We already know that the OLS estimator $\hat{\beta}$ is linear and unbiased (conditionally on X's).
- Let $\tilde{\beta}$ be any other estimator of β such that
  - $\tilde{\beta}$ is linear:
    $$\tilde{\beta} = \sum_{i=1}^{n} c_i Y_i,$$
    where c's depend only on X's.
  - $\tilde{\beta}$ is unbiased:
    $$E\tilde{\beta} = \beta,$$
    where expectation is conditional on X's.
- We need to show that for any such $\tilde{\beta} \neq \hat{\beta}$,
  $$Var\left(\tilde{\beta}\right) > Var\left(\hat{\beta}\right),$$
  where the variance is conditional on X's.

An outline of the proof

1. First, we are going to show that the c's in $\tilde{\beta} = \sum_{i=1}^{n} c_i Y_i$ satisfy $\sum_{i=1}^{n} c_i = 0$ and $\sum_{i=1}^{n} c_i X_i = 1$.

An outline of the proof

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2. Using the results of Step 1, we will show that conditionally on X's, $Cov\left(\tilde{\beta}, \hat{\beta}\right) = Var\left(\hat{\beta}\right)$.

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2. Using the results of Step 1, we will show that conditionally on X's, $Cov\left(\tilde{\beta}, \hat{\beta}\right) = Var\left(\hat{\beta}\right)$.
3. Using the results of Step 2, we will show that conditionally on X's, $Var\left(\tilde{\beta}\right) \geq Var\left(\hat{\beta}\right)$.

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1. First, we are going to show that the c's in $\tilde{\beta} = \sum_{i=1}^{n} c_i Y_i$ satisfy $\sum_{i=1}^{n} c_i = 0$ and $\sum_{i=1}^{n} c_i X_i = 1$.
2. Using the results of Step 1, we will show that conditionally on X's, $Cov\left(\tilde{\beta}, \hat{\beta}\right) = Var\left(\hat{\beta}\right)$.
3. Using the results of Step 2, we will show that conditionally on X's, $Var\left(\tilde{\beta}\right) \geq Var\left(\hat{\beta}\right)$.
4. Lastly, we will show that $Var\left(\tilde{\beta}\right) = Var\left(\hat{\beta}\right)$ if and only if $\tilde{\beta} = \hat{\beta}$.

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$$\begin{aligned} \tilde{\beta} &= \sum_{i=1}^{n} c_i \left(\alpha + \beta X_i + U_i\right) \\ &= \alpha \sum_{i=1}^{n} c_i + \beta \sum_{i=1}^{n} c_i X_i + \sum_{i=1}^{n} c_i U_i. \end{aligned}$$

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- Since this has to be true for any α and β, it follows now that

$$\begin{aligned} \sum_{i=1}^{n} c_i &= 0, \\ \sum_{i=1}^{n} c_i X_i &= 1. \end{aligned}$$

Proof: Step 2

- We have

$$\tilde{\beta} = \beta + \sum_{i=1}^{n} c_i U_i, \text{ with } \sum_{i=1}^{n} c_i = 0, \sum_{i=1}^{n} c_i X_i = 1.$$

$$\hat{\beta} = \beta + \sum_{i=1}^{n} w_i U_i, \text{ with } w_i = \frac{X_i - \bar{X}}{\sum_{j=1}^{n} \left(X_j - \bar{X}\right)^2}.$$

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- Thus,

$$Cov\left(\tilde{\beta}, \hat{\beta}\right) = \sigma^2 \sum_{i=1}^{n} c_i w_i.$$

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Conditionally on X's:

$$Cov\left(\tilde{\beta}, \hat{\beta}\right) = \sigma^2 \sum_{i=1}^{n} c_i w_i \text{ and } w_i = \frac{X_i - \bar{X}}{\sum_{j=1}^{n}\left(X_j - \bar{X}\right)^2}.$$

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&= \frac{\sigma^2}{\sum_{j=1}^{n}\left(X_j - \bar{X}\right)^2} \sum_{i=1}^{n} c_i \left(X_i - \bar{X}\right) \\
&= \frac{\sigma^2}{\sum_{j=1}^{n}\left(X_j - \bar{X}\right)^2} \left(\sum_{i=1}^{n} c_i X_i - \bar{X} \sum_{i=1}^{n} c_i\right) \\
&= \frac{\sigma^2}{\sum_{j=1}^{n}\left(X_j - \bar{X}\right)^2} \left(1 + \bar{X} \cdot 0\right)
\end{aligned}$$

Proof: Step 2

Conditionally on X's:

$$Cov\left(\tilde{\beta}, \hat{\beta}\right) = \sigma^2 \sum_{i=1}^{n} c_i w_i \text{ and } w_i = \frac{X_i - \bar{X}}{\sum_{j=1}^{n}\left(X_j - \bar{X}\right)^2}.$$

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&= Var\left(\hat{\beta}\right).
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- But since $Var\left(\tilde{\beta} - \hat{\beta}\right) \geq 0$,

$$Var\left(\tilde{\beta}\right) - Var\left(\hat{\beta}\right) \geq 0$$

or

$$Var\left(\tilde{\beta}\right) \geq Var\left(\hat{\beta}\right).$$

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- This constant also has to be zero because
$$\begin{aligned} E\tilde{\beta} &= E\hat{\beta} + \text{constant} \\ &= \beta + \text{constant}, \end{aligned}$$
and in order for $\tilde{\beta}$ to be unbiased
$$\text{constant}=0 \text{ or } \tilde{\beta} = \hat{\beta}.$$