Economics 326
Methods of Empirical Research in Economics

Lecture 6: Estimating the variance of errors

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The importance of σ^2

- The variance of $\hat{\beta}$ depends on unknown $\sigma^2 = EU_i^2$:

$$Var\left(\hat{\beta}|X_1,\ldots,X_n\right) = \frac{\sigma^2}{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2}.$$

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$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n \hat{U}_i^2.$$

- It turns out that $\hat{\sigma}^2$ is biased .

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  2. $E\left(U_i | X_1, \dots, X_n\right) = 0$ for all i's,
  3. $E\left(U_i^2 | X_1, \dots, X_n\right) = \sigma^2$ for all i's,

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  4. $E\left(U_i U_j | X_1, \ldots, X_n\right) = 0$ for all $i \neq j$.

An unbiased estimator of σ^2

$$s^2 = \frac{1}{n-2} \sum_{i=1}^{n} \hat{U}_i^2.$$

- In order to construct $\hat{U}_i$, first we need to estimate two parameters α and β :

$$\hat{U}_i = Y_i - \hat{\alpha} - \hat{\beta} X_i,$$

and s^2 adjusts for the estimation of α and β by dividing by $n-2$ instead of n.

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- To show that $Es^2 = \sigma^2$ (unbiasedness), we need to show that

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- We need to express $\hat{U}$'s using U's, since $EU_i^2 = \sigma^2$.

Expansion of $\hat{U}_i$

From

$$\begin{aligned} \hat{U}_i &= Y_i - \hat{\alpha} - \hat{\beta} X_i, \text{ and} \\ \hat{\alpha} &= \bar{Y} - \hat{\beta} \bar{X}, \end{aligned}$$

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we have

$$\hat{U}_i = Y_i - \left(\bar{Y} - \hat{\beta}\bar{X}\right) - \hat{\beta} X_i$$

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$$\begin{aligned} \hat{U}_i &= Y_i - \left(\bar{Y} - \hat{\beta}\bar{X}\right) - \hat{\beta} X_i \\ &= \left(Y_i - \bar{Y}\right) - \hat{\beta}\left(X_i - \bar{X}\right). \end{aligned} \tag{1}$$

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Expansion of $\hat{U}_i$

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we have

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By combining (1) and (2),

$$\hat{U}_i = \left(U_i - \bar{U}\right) - \left(\hat{\beta} - \beta\right) \left(X_i - \bar{X}\right).$$

Expansion of $\sum_{i=1}^{n} \hat{U}_i^2$

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$$\begin{aligned} \hat{U}_i^2 &= \left[(U_i - \bar{U}) - \left(\hat{\beta} - \beta\right)\left(X_i - \bar{X}\right)\right]^2 \\ &= \left(U_i - \bar{U}\right)^2 + \left(\hat{\beta} - \beta\right)^2 \left(X_i - \bar{X}\right)^2 - \\ &\quad -2\left(\hat{\beta} - \beta\right)\left(X_i - \bar{X}\right)\left(U_i - \bar{U}\right). \end{aligned}$$

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Thus,

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\sum_{i=1}^{n} \hat{U}_i^2 &= \sum_{i=1}^{n} (U_i - \bar{U})^2 + \left(\hat{\beta} - \beta\right)^2 \sum_{i=1}^{n} (X_i - \bar{X})^2 \\
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We will show that

- $E\left[\sum_{i=1}^{n} \left(U_i - \bar{U}\right)^2\right] = (n-1)\,\sigma^2$,

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and therefore,

$$E \sum_{i=1}^{n} \hat{U}_i^2 = (n-1)\,\sigma^2 + \sigma^2 - 2\sigma^2$$

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$E \sum_{i=1}^{n} \left(U_i - \bar{U}\right)^2 = (n-1)\sigma^2$

First,

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&= \sum_{i=1}^{n} U_i^2 - \left(\frac{1}{n} \sum_{i=1}^{n} U_i\right) \sum_{i=1}^{n} U_i
\end{aligned}$$

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$E\sum_{i=1}^{n}\left(U_i-\bar{U}\right)^2=(n-1)\sigma^2$

$$\sum_{i=1}^{n}\left(U_i-\bar{U}\right)^2 = \sum_{i=1}^{n}U_i^2-\frac{1}{n}\left(\sum_{i=1}^{n}U_i\right)^2$$

$E \sum_{i=1}^{n} (U_i - \bar{U})^2 = (n-1)\sigma^2$

$$\begin{aligned}\sum_{i=1}^{n} (U_i - \bar{U})^2 &= \sum_{i=1}^{n} U_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} U_i\right)^2 \\ &= \sum_{i=1}^{n} U_i^2 - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} U_i U_j\end{aligned}$$

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&= \sum_{i=1}^{n}U_i^2-\frac{1}{n}\left(\sum_{i=1}^{n}U_i^2+\sum_{i=1}^{n}\sum_{j\neq i}U_iU_j\right).
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$$

$E\sum_{i=1}^{n}(U_i-\bar{U})^2=(n-1)\sigma^2$

$$\begin{aligned}\sum_{i=1}^{n}(U_i-\bar{U})^2 &= \sum_{i=1}^{n}U_i^2-\frac{1}{n}\left(\sum_{i=1}^{n}U_i\right)^2\\ &= \sum_{i=1}^{n}U_i^2-\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}U_iU_j\\ &= \sum_{i=1}^{n}U_i^2-\frac{1}{n}\left(\sum_{i=1}^{n}U_i^2+\sum_{i=1}^{n}\sum_{j\neq i}U_iU_j\right).\end{aligned}$$

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$E \sum_{i=1}^{n} (U_i - \bar{U})^2 = (n-1)\sigma^2$

$$\sum_{i=1}^{n} (U_i - \bar{U})^2 = \sum_{i=1}^{n} U_i^2 - \frac{1}{n} \left(\sum_{i=1}^{n} U_i \right)^2$$

$$= \sum_{i=1}^{n} U_i^2 - \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} U_i U_j$$

$$= \sum_{i=1}^{n} U_i^2 - \frac{1}{n} \left(\sum_{i=1}^{n} U_i^2 + \sum_{i=1}^{n} \sum_{j \neq i} U_i U_j \right).$$

$$E \sum_{i=1}^{n} (U_i - \bar{U})^2 = \sum_{i=1}^{n} E U_i^2 - \frac{1}{n} \left(\sum_{i=1}^{n} E U_i^2 + \sum_{i=1}^{n} \sum_{j \neq i} E U_i U_j \right)$$

$$= \sum_{i=1}^{n} \sigma^2 - \frac{1}{n} \left(\sum_{i=1}^{n} \sigma^2 + 0 \right) = n\sigma^2 - \frac{1}{n} n\sigma^2$$

$E\sum_{i=1}^{n}\left(U_i-\bar{U}\right)^2=(n-1)\sigma^2$

$$\begin{aligned}
\sum_{i=1}^{n}\left(U_i-\bar{U}\right)^2 &= \sum_{i=1}^{n}U_i^2-\frac{1}{n}\left(\sum_{i=1}^{n}U_i\right)^2\\
&= \sum_{i=1}^{n}U_i^2-\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}U_iU_j\\
&= \sum_{i=1}^{n}U_i^2-\frac{1}{n}\left(\sum_{i=1}^{n}U_i^2+\sum_{i=1}^{n}\sum_{j\neq i}U_iU_j\right).
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&= \sum_{i=1}^{n}\sigma^2-\frac{1}{n}\left(\sum_{i=1}^{n}\sigma^2+0\right)=n\sigma^2-\frac{1}{n}n\sigma^2\\
&= (n-1)\,\sigma^2.
\end{aligned}$$

$E\left(\hat{\beta}-\beta\right)^2 \sum_{i=1}^{n}\left(X_i-\bar{X}\right)^2=\sigma^2$

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$$Var\left(\hat{\beta}\right)=\frac{\sigma^2}{\sum_{i=1}^{n}\left(X_i-\bar{X}\right)^2}.$$

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=\ &\frac{1}{\sum_{i=1}^{n}\left(X_i-\bar{X}\right)^2}\left(\sum_{i=1}^{n}\left(X_i-\bar{X}\right)U_i\right)^2.
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$$E\left[\left(\hat{\beta}-\beta\right)\sum_{i=1}^{n}\left(X_i-\bar{X}\right)\left(U_i-\bar{U}\right)\right]$$

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= \; & \frac{1}{\sum_{i=1}^{n}\left(X_i-\bar{X}\right)^2}\left(\sigma^2\sum_{i=1}^{n}\left(X_i-\bar{X}\right)^2\right)=\sigma^2.
\end{aligned}$$

Estimation of the variance of $\hat{\beta}$

- The variance of $\hat{\beta}$ (conditional on X's):

$$Var\left(\hat{\beta}\right) = \frac{\sigma^2}{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2}.$$

It is unknown because σ^2 is unknown.

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- The estimator of σ^2 :

$$s^2 = \frac{1}{n-2}\sum_{i=1}^{n}\hat{U}_i^2 = \frac{1}{n-2}\sum_{i=1}^{n}\left(Y_i - \hat{\alpha} - \hat{\beta}X_i\right)^2.$$

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- The standard errorof β:

$$SE\left(\hat{\beta}\right) = \sqrt{\widehat{Var}\left(\hat{\beta}\right)} = \sqrt{\frac{s^2}{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2}}.$$