

Economics 326
Methods of Empirical Research in Economics

Lecture 8: Hypothesis testing (Part 1)

Vadim Marmer
University of British Columbia

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Hypothesis testing

- ▶ Hypothesis testing is one of a fundamental problems in statistics.
- ▶ A hypothesis is (usually) an assertion about the unknown population parameters such as β_1 in $Y_i = \beta_0 + \beta_1 X_i + U_i$.
- ▶ Using the data, the econometrician has to determine whether an assertion is true or false.
- ▶ Example: Phillips curve:

$$\text{Unemployment}_t = \beta_0 + \beta_1 \text{Inflation}_t + U_t.$$

In this example, we are interested in testing if $\beta_1 = 0$ (no Phillips curve) against $\beta_1 < 0$ (Phillips curve).

Assumptions

- ▶ β_1 is unknown, and we have to rely on its OLS estimator $\hat{\beta}_1$.
- ▶ We need to know the distribution of $\hat{\beta}_1$ or of its certain functions.
- ▶ We will assume that the assumptions of the Normal Classical Linear Regression model are satisfied:
 1. $Y_i = \beta_0 + \beta_1 X_i + U_i, \quad i = 1, \dots, n.$
 2. $E(U_i | X_1, \dots, X_n) = 0$ for all i 's.
 3. $E(U_i^2 | X_1, \dots, X_n) = \sigma^2$ for all i 's.
 4. $E(U_i U_j | X_1, \dots, X_n) = 0$ for all $i \neq j$.
 5. U 's are jointly normally distributed conditional on X 's.
- ▶ Recall that, in this case, conditionally on X 's:

$$\hat{\beta}_1 \sim N(\beta_1, \text{Var}(\hat{\beta}_1)), \text{ where } \text{Var}(\hat{\beta}_1) = \frac{\sigma^2}{\sum_{i=1}^n (X_i - \bar{X})^2}.$$

Null and alternative hypotheses

- ▶ Usually, we have two competing hypotheses, and we want to draw a conclusion, based on the data, as to which of the hypotheses is true.
- ▶ Null hypothesis , denoted as H_0 : A hypothesis that is held to be true, unless the data provides a sufficient evidence against it.
- ▶ Alternative hypothesis, denoted as H_1 : A hypothesis against which the null is tested. It is held to be true if the null is found false.
- ▶ Usually, the econometrician has to carry the "burden of proof," and the case that he is interested in is stated as H_1 .
- ▶ The econometrician has to prove that his assertion (H_1) is true by showing that the data rejects H_0 .
- ▶ The two hypotheses must be disjoint: it should be the case that either H_0 is true or H_1 but never together simultaneously.

Decision rule

- ▶ The econometrician has to choose between H_0 and H_1 .
- ▶ The decision rule that leads the econometrician to reject or not to reject H_0 is based on a test statistic, which is a function of the data $\{(Y_i, X_i) : i = 1, \dots, n\}$.
- ▶ Usually, one rejects H_0 if the test statistic falls into a critical region. A critical region is constructed by taking into account the probability of making a wrong decision.

Errors

- ▶ There are two types of errors that the econometrician can make:

		Truth	
		H_0	H_1
Decision	H_0	✓	Type II error
	H_1	Type I error	✓

- ▶ Type I error is the error of rejecting H_0 when H_0 is true.
- ▶ The probability of Type I error is denoted by α and called significance level or size of a test :

$$P(\text{Type I error}) = P(\text{reject } H_0 | H_0 \text{ is true}) = \alpha.$$

- ▶ Type II error is the error of not rejecting H_0 when H_1 is true.
- ▶ Power of a test:

$$1 - P(\text{Type II error}) = 1 - P(\text{Accept } H_0 | H_0 \text{ is false}).$$

Errors

- ▶ The decision rule depends on a test statistic T .
- ▶ The real line is split into two regions: acceptance region and rejection region (critical region).
- ▶ When T is in the acceptance region, we accept H_0 (and risk making a Type II error).
- ▶ When T is in the rejection (critical) region, we reject H_0 (and risk making a Type I error).
- ▶ Unfortunately, the probabilities of Type I and II errors are inversely related. By decreasing the probability of Type I error α , one makes the critical region smaller, which increases the probability of the Type II error. Thus it is impossible to make both errors arbitrary small.
- ▶ By convention, α is chosen to be a small number, for example, $\alpha = 0.01, 0.05$, or 0.10 . (This is in agreement with the econometrician carrying the burden of proof).

Steps

- ▶ The following are the steps of the hypothesis testing:
 1. Specify H_0 and H_1 .
 2. Choose the significance level α .
 3. Define a decision rule (critical region).
 4. Perform the test using the data: given the data compute the test statistic and see if it falls into the critical region.
- ▶ The decision depends on the significance level α : larger values of α correspond to bigger critical regions (probability of Type I error is larger).
- ▶ It is easier to reject the null for larger values of α .
- ▶ p -value : Given the data, the smallest significance level at which the null can be rejected.

Two-sided tests

- ▶ For $Y_i = \beta_0 + \beta_1 X_i + U_i$, consider testing

$$H_0 : \beta_1 = \beta_{1,0},$$

against

$$H_1 : \beta_1 \neq \beta_{1,0}.$$

- ▶ β_1 is the true unknown value of the slope parameter.
- ▶ $\beta_{1,0}$ is a known number specified by the econometrician. (For example $\beta_{1,0}$ is zero if you want to test $\beta_1 = 0$).
- ▶ Such a test is called two-sided because the alternative hypothesis H_1 does not specify in which direction β_1 can deviate from the asserted value $\beta_{1,0}$.

Two-sided tests when σ^2 is known (infeasible test)

- ▶ Suppose for a moment that σ^2 is known.
- ▶ Consider the following test statistic:

$$T = \frac{\hat{\beta}_1 - \beta_{1,0}}{\sqrt{\text{Var}(\hat{\beta}_1)}}, \text{ where } \text{Var}(\hat{\beta}_1) = \frac{\sigma^2}{\sum_{i=1}^n (X_i - \bar{X})^2}.$$

- ▶ Consider the following decision rule (test):

$$\text{Reject } H_0 : \beta_1 = \beta_{1,0} \text{ when } |T| > z_{1-\alpha/2},$$

where $z_{1-\alpha/2}$ is the $(1 - \alpha/2)$ quantile of the standard normal distribution (critical value).

Test validity and power

- ▶ We need to establish that:

1. The test is valid, where the validity of a test means that it has correct size or $P(\text{Type I error}) = \alpha$:

$$P(|T| > z_{1-\alpha/2} | \beta_1 = \beta_{1,0}) = \alpha.$$

2. The test has power: when $\beta_1 \neq \beta_{1,0}$ (H_0 is false), the test rejects H_0 with probability that exceeds α :

$$P(|T| > z_{1-\alpha/2} | \beta_1 \neq \beta_{1,0}) > \alpha.$$

- ▶ We want $P(|T| > z_{1-\alpha/2} | \beta_1 \neq \beta_{1,0})$ to be as large as possible.
- ▶ Note that $P(|T| > z_{1-\alpha/2} | \beta_1 \neq \beta_{1,0})$ depends on the true value β_1 .

The distribution of T when σ^2 is known (infeasible test)

- Write

$$\begin{aligned} T &= \frac{\hat{\beta}_1 - \beta_{1,0}}{\sqrt{\text{Var}(\hat{\beta}_1)}} = \frac{\hat{\beta}_1 - \beta_1 + \beta_1 - \beta_{1,0}}{\sqrt{\text{Var}(\hat{\beta}_1)}} \\ &= \frac{\hat{\beta}_1 - \beta_1}{\sqrt{\text{Var}(\hat{\beta}_1)}} + \frac{\beta_1 - \beta_{1,0}}{\sqrt{\text{Var}(\hat{\beta}_1)}}. \end{aligned}$$

- Under our assumptions and conditionally on X 's:

$$\hat{\beta}_1 \sim N(\beta_1, \text{Var}(\hat{\beta}_1)), \text{ or } \frac{\hat{\beta}_1 - \beta_1}{\sqrt{\text{Var}(\hat{\beta}_1)}} \sim N(0, 1).$$

- We have that conditionally on X 's: $T \sim N\left(\frac{\beta_1 - \beta_{1,0}}{\sqrt{\text{Var}(\hat{\beta}_1)}}, 1\right)$.

Size when σ^2 is known (infeasible test)

- ▶ We have that

$$T \sim N\left(\frac{\beta_1 - \beta_{1,0}}{\sqrt{\text{Var}(\hat{\beta}_1)}}, 1\right).$$

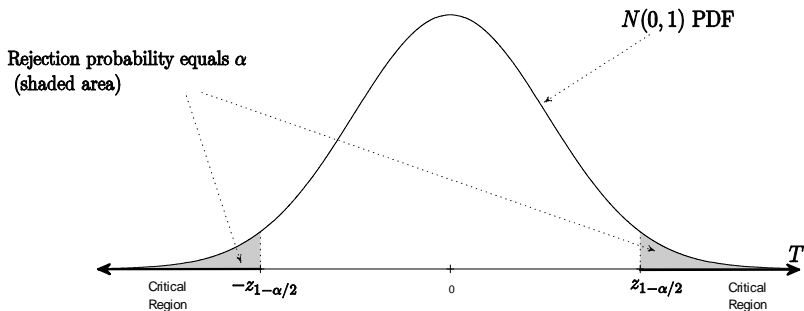
- ▶ When $H_0 : \beta_1 = \beta_{1,0}$ is true, $T \stackrel{H_0}{\sim} N(0, 1)$.
- ▶ We reject H_0 when

$$|T| > z_{1-\alpha/2} \Leftrightarrow T > z_{1-\alpha/2} \text{ or } T < -z_{1-\alpha/2}.$$

- ▶ Let $Z \sim N(0, 1)$.

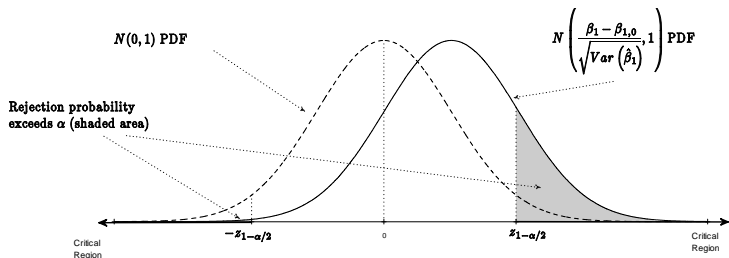
$$\begin{aligned} P(\text{Reject } H_0 | H_0 \text{ is true}) &= P(Z > z_{1-\alpha/2}) + P(Z < -z_{1-\alpha/2}) \\ &= \alpha/2 + \alpha/2 = \alpha \end{aligned}$$

The distribution of T when σ^2 is known (infeasible test)



Power of the test when σ^2 is known (infeasible test)

- ▶ Under H_1 , $\beta_1 - \beta_{1,0} \neq 0$ and the distribution of T is not centered zero: $T \sim N\left(\frac{\beta_1 - \beta_{1,0}}{\sqrt{\text{Var}(\hat{\beta}_1)}}, 1\right)$.
- ▶ When $\beta_1 - \beta_{1,0} > 0$:



- ▶ Rejection probability exceeds α under H_1 : power increases with the distance from H_0 ($|\beta_{1,0} - \beta_1|$) and decreases with $\text{Var}(\hat{\beta}_1)$.