

Economics 326

Methods of Empirical Research in Economics

Lecture 18: The asymptotic variance of OLS and
heteroskedasticity

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Asymptotic normality

- ▶ In the previous lecture, we showed that when the data are iid and the regressors are exogenous:

$$\begin{aligned} Y_i &= \beta_0 + \beta_1 X_i + U_i, \\ EU_i &= E(X_i U_i) = 0, \end{aligned}$$

the OLS estimator of β_1 is asymptotically normal:

$$\begin{aligned} \sqrt{n} (\hat{\beta}_{1,n} - \beta_1) &\rightarrow_d N(0, V), \\ V &= \frac{E\left((X_i - EX_i)^2 U_i^2\right)}{(\text{Var}(X_i))^2}. \end{aligned}$$

- ▶ For the purpose of hypothesis testing, we need to obtain a **consistent** estimator of the asymptotic variance V :

$$\hat{V}_n \rightarrow_p V.$$

Homoskedastic errors

- ▶ Let's assume that the errors are **homoskedastic**:

$$E(U_i^2 | X_i) = \sigma^2 \text{ for all } X_i\text{'s.}$$

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$$E\left((X_i - EX_i)^2 U_i^2\right) = EE\left[(X_i - EX_i)^2 U_i^2 | X_i\right]$$

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Homoskedastic errors

- ▶ Thus, when the errors are homoskedastic with $EU_i^2 = \sigma^2$,

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- ▶ Let $\hat{U}_i = Y_i - \hat{\beta}_{0,n} - \hat{\beta}_{1,n}X_i$, where $\hat{\beta}_{0,n}$ and $\hat{\beta}_{1,n}$ are the OLS estimators of β_0 and β_1 .

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- ▶ A consistent estimator for the asymptotic variance can be constructed by using the **Method of Moments**.

$$\hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^n \hat{U}_i^2,$$

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$$\widehat{\text{Var}}(X_i) = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2,$$

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$$\widehat{\text{Var}}(X_i) = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2, \text{ and}$$

$$\hat{V}_n = \frac{\hat{\sigma}_n^2}{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2}.$$

Homoskedastic errors

$$\hat{V}_n = \frac{\hat{\sigma}_n^2}{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2}, \quad \hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^n \hat{U}_i^2, \quad \hat{U}_i = Y_i - \hat{\beta}_{0,n} - \hat{\beta}_{1,n} X_i.$$

- ▶ When proving the consistency of OLS in Lecture 16, we showed that

$$\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2 \rightarrow_p \text{Var}(X_i),$$

and to establish $\hat{V}_n \rightarrow_p V$, we need to show that $\hat{\sigma}_n^2 \rightarrow_p \sigma^2$.

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$$\hat{V}_n = \frac{\hat{\sigma}_n^2}{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2}, \quad \hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^n \hat{U}_i^2, \quad \hat{U}_i = Y_i - \hat{\beta}_{0,n} - \hat{\beta}_{1,n} X_i.$$

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- ▶ Note that the LLN **cannot** be applied directly to

$$\frac{1}{n} \sum_{i=1}^n \hat{U}_i^2$$

because \hat{U}_i 's are not iid: they are **dependent** through $\hat{\beta}_{0,n}$ and $\hat{\beta}_{1,n}$.

Proof of $\hat{\sigma}_n^2 \rightarrow_p \sigma_n^2$

- ▶ First, write

$$\hat{U}_i = Y_i - \hat{\beta}_{0,n} - \hat{\beta}_{1,n}X_i$$

Proof of $\hat{\sigma}_n^2 \rightarrow_p \sigma_n^2$

► First, write

$$\begin{aligned}\hat{U}_i &= Y_i - \hat{\beta}_{0,n} - \hat{\beta}_{1,n}X_i \\ &= (\beta_0 + \beta_1X_i + U_i) - \hat{\beta}_{0,n} - \hat{\beta}_{1,n}X_i\end{aligned}$$

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- ▶ Now,

$$\hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^n \hat{U}_i^2$$

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- ▶ Now,

$$\hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^n \hat{U}_i^2 = \frac{1}{n} \sum_{i=1}^n (U_i - (\hat{\beta}_{0,n} - \beta_0) - (\hat{\beta}_{1,n} - \beta_1) X_i)^2.$$

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► By the LLN,

$$\frac{1}{n} \sum_{i=1}^n U_i^2 \rightarrow_p EU_i^2 = \sigma^2.$$

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- ▶ By the LLN,

$$\frac{1}{n} \sum_{i=1}^n U_i^2 \rightarrow_p EU_i^2 = \sigma^2.$$

- ▶ Because $\hat{\beta}_{0,n}$ and $\hat{\beta}_{1,n}$ are consistent,

$$\hat{\beta}_{0,n} - \beta_0 \rightarrow_p 0 \text{ and } \hat{\beta}_{1,n} - \beta_1 \rightarrow_p 0.$$

Homoskedastic errors

- ▶ Thus, when the errors are homoskedastic,

$$\hat{V}_n = \frac{\hat{\sigma}_n^2}{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2}, \text{ with } \hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^n \hat{U}_i^2,$$

is a consistent estimator of $V = \frac{\sigma^2}{\text{Var}(X_i)}$.

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- ▶ Note that

$$s^2 = \frac{1}{n-2} \sum_{i=1}^n \hat{U}_i^2 \rightarrow_p \sigma^2,$$

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is also a consistent estimator of $V = \frac{\sigma^2}{\text{Var}(X_i)}$.

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is also a consistent estimator of $V = \frac{\sigma^2}{\text{Var}(X_i)}$.

- ▶ This version has an advantage over the one with $\hat{\sigma}_n^2$: in addition to being consistent, s^2 is also an unbiased estimator of σ^2 if the regressors are strongly exogenous.

Homoskedastic errors: Asymptotic approximation

- ▶ Recall that $\sqrt{n} (\hat{\beta}_{1,n} - \beta_1) \rightarrow_d N(0, V)$ is used as the following **approximation**:

$$\hat{\beta}_{1,n} \overset{a}{\sim} N\left(\beta_1, \frac{V}{n}\right),$$

where $\overset{a}{\sim}$ denotes approximately in large samples.

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$$\frac{\hat{V}_n}{n} = \frac{s^2}{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2} \frac{1}{n}$$

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$$\frac{\hat{V}_n}{n} = \frac{s^2}{\sum_{i=1}^n (X_i - \bar{X}_n)^2}$$

- ▶ Thus, in the case of homoskedastic errors we have the following asymptotic approximation:

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- ▶ In finite samples, we have the same result **exactly**, when the regressors are **strongly exogenous** and the errors are **normal**.

Asymptotic T -test

- ▶ Consider testing $H_0 : \beta_1 = \beta_{1,0}$ vs $H_1 : \beta_1 \neq \beta_{1,0}$.

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- ▶ Consider the behavior of T statistic under $H_0 : \beta_1 = \beta_{1,0}$.
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we have that

$$T = \frac{(\hat{\beta}_{1,n} - \beta_{1,0})}{\sqrt{\hat{V}_n/n}}$$

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$$T = \frac{(\hat{\beta}_{1,n} - \beta_{1,0})}{\sqrt{\hat{V}_n/n}} = \frac{\sqrt{n} (\hat{\beta}_{1,n} - \beta_{1,0})}{\sqrt{\hat{V}_n}}$$

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- ▶ Asymptotically, the variance of the OLS estimator is known - we behave as if the variance was known.

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A heteroskedasticity consistent (HC) estimator of the asymptotic variance of OLS

- ▶ In the case of heteroskedastic errors, a consistent estimator of

$V = \frac{E((X_i - EX_i)^2 U_i^2)}{(\text{Var}(X_i))^2}$ can be constructed as follows:

$$\hat{V}_n^{HC} = \frac{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2 \hat{U}_i^2}{\left(\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2\right)^2}.$$

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- ▶ One can show that $\hat{V}_n^{HC} \rightarrow_p V$ when the errors are heteroskedastic or homoskedastic.
- ▶ We have the following asymptotic approximation:

$$\hat{\beta}_{1,n} \overset{a}{\sim} N\left(\beta_1, \frac{\hat{V}_n^{HC}}{n}\right),$$

and the standard errors can be computed as

$$SE(\hat{\beta}_{1,n}) = \sqrt{\hat{V}_n^{HC} / n}.$$

HC variance estimation in Stata

- ▶ In Stata, the HC estimator of standard errors can be obtained by adding the option `robust` to the regression command:

```
. regress liver alcohol, robust
```

		Robust					
	liver	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]	
	alcohol	3.586388	.550515	6.51	0.000	2.434147	4.73863
	_cons	10.85482	2.119993	5.12	0.000	6.417625	15.29202

- ▶ Compare with the non-HC standard errors based on \hat{V}_n :

```
. regress liver alcohol
```

	liver	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]	
	alcohol	3.586388	.7541228	4.76	0.000	2.007991	5.164786
	_cons	10.85482	2.802408	3.87	0.001	4.989313	16.72033
