

VECTOR AND MATRIX DIFFERENTIATION

1 Derivatives of Ax

Let $a \in R^n$, $x \in R^n$ (all vectors are column vectors). Then

$$\begin{aligned} \frac{\partial(a'x)}{\partial x} &= \begin{pmatrix} \frac{\partial(a'x)}{\partial x_1} \\ \vdots \\ \frac{\partial(a'x)}{\partial x_n} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial(a_1x_1 + \dots + a_nx_n)}{\partial x_1} \\ \vdots \\ \frac{\partial(a_1x_1 + \dots + a_nx_n)}{\partial x_n} \end{pmatrix} \\ &= \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \\ &= a. \end{aligned}$$

$$\begin{aligned} \frac{\partial(a'x)}{\partial x'} &= \left(\frac{\partial(a'x)}{\partial x_1} \quad \dots \quad \frac{\partial(a'x)}{\partial x_n} \right) \\ &= (a_1 \quad \dots \quad a_n) \\ &= a'. \end{aligned}$$

Let A be a $m \times n$ matrix,

$$A = \begin{pmatrix} a'_1 \\ \vdots \\ a'_m \end{pmatrix},$$

where $a_j \in R^n$ for $j = 1, \dots, m$.

$$\begin{aligned} \frac{\partial(Ax)}{\partial x'} &= \begin{pmatrix} \frac{\partial(a'_1x)}{\partial x'} \\ \vdots \\ \frac{\partial(a'_mx)}{\partial x'} \end{pmatrix} \\ &= \begin{pmatrix} a'_1 \\ \vdots \\ a'_m \end{pmatrix} \\ &= A. \end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial(x'A')}{\partial x} &= \left(\frac{\partial(x'a_1)}{\partial x} \quad \cdots \quad \frac{\partial(x'a_m)}{\partial x} \right) \\ &= \left(a_1 \quad \cdots \quad a_m \right) \\ &= A'.\end{aligned}$$

Let $\alpha \in R^r$, and $x = x(\alpha)$. Then

$$\frac{\partial x}{\partial \alpha'} = \begin{pmatrix} \frac{\partial x_1}{\partial \alpha'} \\ \vdots \\ \frac{\partial x_n}{\partial \alpha'} \end{pmatrix},$$

an $n \times r$ matrix.

Let $x = x(\alpha)$. Then

$$\begin{aligned}\frac{\partial(a'x)}{\partial \alpha} &= \begin{pmatrix} \frac{\partial(a_1x_1 + \cdots + a_nx_n)}{\partial \alpha_1} \\ \vdots \\ \frac{\partial(a_1x_1 + \cdots + a_nx_n)}{\partial \alpha_r} \end{pmatrix} \\ &= \begin{pmatrix} a_1 \frac{\partial x_1}{\partial \alpha_1} + \cdots + a_n \frac{\partial x_n}{\partial \alpha_1} \\ \vdots \\ a_1 \frac{\partial x_1}{\partial \alpha_r} + \cdots + a_n \frac{\partial x_n}{\partial \alpha_r} \end{pmatrix} \\ &= \frac{\partial x'}{\partial \alpha} a,\end{aligned}$$

an r -vector, and

$$\frac{\partial(a'x)}{\partial \alpha'} = a' \frac{\partial x}{\partial \alpha'}.$$

Next,

$$\begin{aligned}\frac{\partial(Ax)}{\partial \alpha'} &= \begin{pmatrix} \frac{\partial(a'_1x)}{\partial \alpha'} \\ \vdots \\ \frac{\partial(a'_mx)}{\partial \alpha'} \end{pmatrix} \\ &= \begin{pmatrix} a'_1 \frac{\partial x}{\partial \alpha'} \\ \vdots \\ a'_m \frac{\partial x}{\partial \alpha'} \end{pmatrix} \\ &= A \frac{\partial x}{\partial \alpha'}.\end{aligned}$$

Note that

$$\frac{\partial x}{\partial x'} = I_n.$$

2 Derivatives of $z'Ax$

Let $x \in R^n$, $z \in R^m$, and A is $m \times n$. Define $c = A'z$, a n -vector. Then,

$$\begin{aligned} \frac{\partial(z'Ax)}{\partial x} &= \frac{\partial(c'x)}{\partial x} \\ &= A'z. \end{aligned}$$

Next,

$$\frac{\partial(z'Ax)}{\partial z} = Ax.$$

Let $x = x(\alpha)$, and $z = z(\alpha)$. Then

$$\begin{aligned} \frac{\partial(z'Ax)}{\partial \alpha} &= \frac{\partial z'}{\partial \alpha} \frac{\partial(z'Ax)}{\partial z} + \frac{\partial x'}{\partial \alpha} \frac{\partial(z'Ax)}{\partial x} \\ &= \frac{\partial z'}{\partial \alpha} Ax + \frac{\partial x'}{\partial \alpha} A'z. \end{aligned}$$

Next, if $m = n$,

$$\begin{aligned} \frac{\partial(x'Ax)}{\partial x} &= Ax + A'x \\ &= (A + A')x. \end{aligned}$$

If A is symmetric,

$$\frac{\partial(x'Ax)}{\partial x} = 2Ax.$$

If $x = x(\alpha)$, A is symmetric and independent of α ,

$$\frac{\partial(x'Ax)}{\partial \alpha} = 2 \frac{\partial x'}{\partial \alpha} Ax.$$

The derivative of $z'Ax$ with respect to A is given by

$$\begin{aligned} \frac{\partial(z'Ax)}{\partial A} &= \begin{pmatrix} \frac{\partial(z'Ax)}{\partial A_{11}} & \cdots & \frac{\partial(z'Ax)}{\partial A_{1n}} \\ \cdots & \cdots & \cdots \\ \frac{\partial(z'Ax)}{\partial A_{m1}} & \cdots & \frac{\partial(z'Ax)}{\partial A_{mn}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial(\sum_{i=1}^m \sum_{j=1}^n z_i A_{ij} x_j)}{\partial A_{11}} & \cdots & \frac{\partial(\sum_{i=1}^m \sum_{j=1}^n z_i A_{ij} x_j)}{\partial A_{1n}} \\ \cdots & \cdots & \cdots \\ \frac{\partial(\sum_{i=1}^m \sum_{j=1}^n z_i A_{ij} x_j)}{\partial A_{m1}} & \cdots & \frac{\partial(\sum_{i=1}^m \sum_{j=1}^n z_i A_{ij} x_j)}{\partial A_{mn}} \end{pmatrix} \\ &= \begin{pmatrix} z_1 x_1 & \cdots & z_1 x_n \\ \cdots & \cdots & \cdots \\ z_m x_1 & \cdots & z_m x_n \end{pmatrix} \\ &= zx'. \end{aligned}$$

3 GMM first-order conditions

Let $z \in R^l$, $x \in R^k$, $b \in R^k$, $y \in R$, and W is $l \times l$ and symmetric. Then

$$\begin{aligned}
 \frac{\partial}{\partial b} \left[(z(y - x'b))' W (z(y - x'b)) \right] &= 2 \frac{\partial (z(y - x'b))'}{\partial b} W (z(y - x'b)) \\
 &= -2 \frac{\partial (zx'b)'}{\partial b} W (z(y - x'b)) \\
 &= -2 \frac{\partial (b'xz')}{\partial b} W (z(y - x'b)) \\
 &= -2xz'W (zy - zx'b).
 \end{aligned}$$

Similarly, suppose that Z is $n \times l$, X is $n \times k$, b is $k \times 1$, Y is $n \times 1$, and W is $l \times l$. Then

$$\begin{aligned}
 \frac{\partial}{\partial b} \left[(Z'(Y - Xb))' W (Z'(Y - Xb)) \right] &= 2 \frac{\partial (Z'(Y - Xb))'}{\partial b} W (Z'(Y - Xb)) \\
 &= -2 \frac{\partial (Z'Xb)'}{\partial b} W (Z'(Y - Xb)) \\
 &= -2 \frac{\partial (b'X'Z)}{\partial b} W (Z'(Y - Xb)) \\
 &= -2X'ZW (Z'Y - Z'Xb).
 \end{aligned}$$

4 ML first-order conditions

Suppose x is a k -vector, and Σ is $k \times k$, symmetric and positive definite. Define

$$Q(\Sigma) = -\log |\Sigma| - x'\Sigma^{-1}x.$$

We use the fact that

$$\frac{\partial \log |\Sigma|}{\partial \Sigma} = \Sigma^{-1},$$

and, hence,

$$\begin{aligned}
 -\frac{\partial \log |\Sigma|}{\partial \Sigma^{-1}} &= \frac{\partial \log |\Sigma^{-1}|}{\partial \Sigma^{-1}} \\
 &= \Sigma.
 \end{aligned}$$

Next,

$$\frac{\partial (x'\Sigma^{-1}x)}{\partial \Sigma^{-1}} = xx'.$$

Therefore, the first-order conditions are given by

$$\Sigma = xx'.$$