Non-Parametric Bounds for Non-Convex Preferences*

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Abstract

Choices from linear budget sets are often used to recover consumer's preferences. The classic method uses revealed preference theory to construct non-parametric bounds on the indifference curve that passes through a given bundle. We show that these bounds do not apply to non-convex preferences, and therefore may lead to erroneous predictions and welfare analysis. We suggest an alternative that is based solely on the assumption of monotonicity of preferences.

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1 Introduction

Recovery of consumers' preferences, and in particular their risk attitudes, plays an important role in financial, health and insurance markets. Varian (1982) suggests a non-parametric recovery method that partially identifies the preferences of a consistent decision maker (henceforth, DM) by constructing, for every bundle, upper and lower bounds on the indifference curve that passes through this bundle. In this short paper, we wish to draw attention to the assumption of convexity of preferences implicitly invoked when using this method.

Indeed, this restriction does not appear in the statement of Varian's suggested method (Fact 5) and, despite being a textbook material and providing the foundation for partial identification of preferences, this issue was never discussed in the literature.

We introduce two examples that demonstrate that if a data set is generated by a DM who correctly maximizes a non-convex preference relation, the underlying indifference curves may not respect the non-parametric bounds suggested in Varian (1982). These examples are used to clarify the technical issue that causes this discrepancy. Moreover, we find this exclusion to be unwarranted, in particular in the context of prediction and welfare analysis in domains where non-convex preferences are crucial and frequent (e.g. risk, ambiguity and other-regarding preferences).

Hence, we provide an alternative approach where the upper and lower bounds on the indifference curve that passes through a given bundle are constructed using only the assumption of monotonicity of preferences. In domains where bundles are composed of goods, this alternative approach is shown to be looser but more reliable than the original approach for prediction and welfare analysis.

2 Varian's Fact 5

2.1 Preliminaries

Consider a DM who chooses bundles $x^i \in \Re^K_+ \quad (i = \{1, \ldots, n\})$ from linear budgets $\left\{x : p^i x \leq p^i x^i,\, p^i \in \Re^K_{++}\right\}$. Let $D = \left\{(p^i, x^i)_{i=1}^n\right\}$ be a finite data set, where x^i is the chosen bundle at prices p^i. The ranking (preference) information encoded in the observed choices is summarized by the following binary relations.

Definition 1. Let $D = \left\{(p^i, x^i)_{i=1}^n\right\}$. An observed bundle $x^i \in \Re^K_+$ is

1. *directly revealed preferred* to a bundle $x \in \Re^K_+$, denoted $x^i R^0_D x$, if $p^i x^i \geq p^i x$.

2. *strictly directly revealed preferred* to a bundle $x \in \Re^K_+$, denoted $x^i P^0_D x$, if $p^i x^i > p^i x$.

3. *revealed preferred* to a bundle $x \in \Re^K_+$, denoted $x^i R_D x$, if there exists a sequence of observed bundles $\left(x^j, x^k, \ldots, x^m\right)$ such that $x^i R^0_D x^j, x^j R^0_D x^k, \ldots, x^m R^0_D x$.

4. *strictly revealed preferred* to a bundle $x \in \Re^K_+$, denoted $x^i P_D x$, if there exists a sequence of observed bundles $\left(x^j, x^k, \ldots, x^m\right)$ such that $x^i R^0_D x^j, x^j R^0_D x^k, \ldots, x^m R^0_D x$ and at least one of them is strict.

The data is said to be consistent if it satisfies the General Axiom of Revealed Preference.

Definition 2. Data set D satisfies the *General Axiom of Revealed Preference (GARP)* if for every pair of observed bundles, $x^i R_D x^j$ implies not $x^j P^0_D x^i$.

The following definition relates the revealed preference information implied by observed choices to the ranking induced by utility maximization.

Definition 3. A utility function $u : \Re^K_+ \to \Re$ *rationalizes* a data set D, if for every observed bundle $x^i \in \Re^K_+$, $u(x^i) \geq u(x)$ for all x such that $x^i R^0_D x$. We say that D is *rationalizable* if such $u(\cdot)$ exists.

Afriat's celebrated theorem provides tight conditions for the rationalizability of a finite data set.

Theorem. (Afriat, 1967) *The following conditions are equivalent:*

1. There exists a non-satiated utility function that rationalizes the data.

2. The data satisfies GARP.

3. There exists a non-satiated, continuous, concave and monotone utility function that rationalizes the data.

Proof. See Afriat (1967); Diewert (1973); Varian (1982); Teo and Vohra (2003); Fostel et al. (2004); Geanakoplos (2013). □

2.2 Bounding the Indifference Curve

Assume that D satisfies GARP. The following definitions follow Varian (1982).

Definition 4. $P_u(x) \equiv \{x' : u(x') > u(x)\}$ is the *strictly upper contour set* of a bundle $x \in \Re_+^K$ given a utility function $u(x)$.

Next, Varian (1982) defines, for a given unobserved bundle x, the set of normalized prices at which x may be chosen such that the augmented data set still satisfies GARP.

Definition 5. Suppose $x \in \Re_+^K$ is an unobserved bundle, then

$$S(x) = \{p \,|\, \{(p, x)\} \cup D \,\text{satisfies GARP and}\, px = 1\}$$

Varian (1982) notes (p. 950) that Afriat's theorem implies that $S(x)$ is nonempty for all $x \in \Re_+^K$ since there exists a concave utility function that rationalizes the data and therefore there exists a supporting price p for every x. For every unobserved bundle x, Varian (1982) employs $S(x)$ to construct lower and upper bounds on the strictly upper contour set through x, using the following definitions.

Definition 6. For every unobserved bundle $x \in \Re_{+}^{K}$:

1. The *revealed worse set* is $RW(x) \equiv \left\{ x' \left| \forall p \in S(x), xP_{D\cup\{p,x\}}x' \right. \right\}$.

2. The *not revealed worse set,* denoted by $NRW(x)$, is the complement of $RW(x)$.

3. The *revealed preferred set* is $RP(x) \equiv \left\{ x' \left| \forall p \in S(x'), x'P_{D\cup\{p,x'\}}x \right. \right\}$.[1]

In Fact 5, Varian (1982, page 953) states: "let $u(x)$ be any utility function that rationalizes the data. Then for all (unobserved bundles - HPZ) x, $RP(x) \subset P_u(x) \subset NRW(x)$". This may be understood as if given a data set that satisfies GARP and a utility function that rationalizes these data, every indifference curve through a given unobserved bundle must be bounded between the revealed worse set and the revealed preferred set of this bundle. In the following section we provide two counter-examples.

3 Two Counter Examples

3.1 Textbook Example

Assume the DM's non-convex preferences are represented by the utility function

$$u(x,y) = \begin{cases} x^3 y & if \quad x \geq y \\ xy^3 & if \quad x < y \end{cases} \tag{3.1}$$

Denote the price of the first good by p_x, the price of the second good by p_y and the DM's income by I.[2] Suppose that the DM faces two problems - one where

[1] Denote by $CM(D,x)$ the convex hull of all the bundles that are revealed preferred to x by D or weakly monotonically greater than bundles that are revealed preferred to x by D. Varian (1982) proves that $CM(D,x)$ is a subset of $RP(x)$ while Knoblauch (1992) shows that $RP(x)$ is a subset of the closure of $CM(D,x)$.

[2]

$$(x,y)^d(p_x,p_y,I) = \begin{cases} (\frac{I}{4p_x}, \frac{3I}{4p_y}) & if \quad \frac{p_x}{p_y} > 1 \\ \left\{ (\frac{I}{4p_x}, \frac{3I}{4p_y}), (\frac{3I}{4p_x}, \frac{I}{4p_y}) \right\} & if \quad \frac{p_x}{p_y} = 1 \\ (\frac{3I}{4p_x}, \frac{I}{4p_y}) & if \quad \frac{p_x}{p_y} < 1 \end{cases}.$$

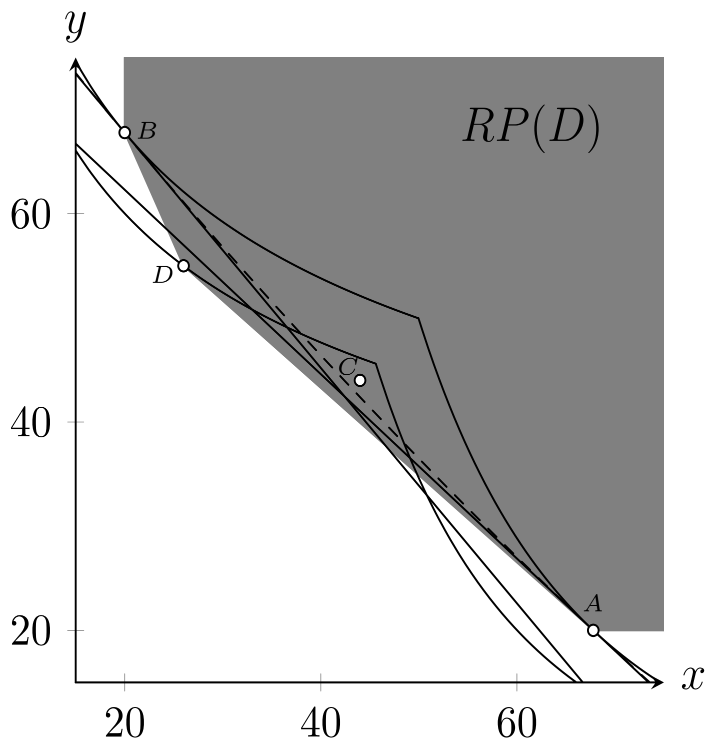


Figure 3.1: Textbook example

the prices are $(p_x^1, p_y^1) = (1, 1.13)$ and $I = 80$ and the other where the prices are $(p_x^2, p_y^2) = (1.13, 1)$ and $I = 80$. Figure 3.1 illustrates the two problems, the DM's optimal choices (bundles A and B, respectively) and the indifference curve that passes through those bundles (the smooth curve through bundles A and B).

Naturally, $u(x, y)$ rationalizes the data. Therefore, Afriat's theorem guarantees that these choices can be rationalized by a continuous, monotone and concave utility function, although the choices were generated by non-convex preferences. Figure 3.1 demonstrates that the utility function $v(x, y) = x^{0.9} + y^{0.9}$ (which is concave and nicely behaved) can rationalize the DM's choices (the dashed curve through bundles A and B).[3] However, u and v may

[3]In fact, this function even preserves the DM's (unobserved) indifference between the two chosen bundles.

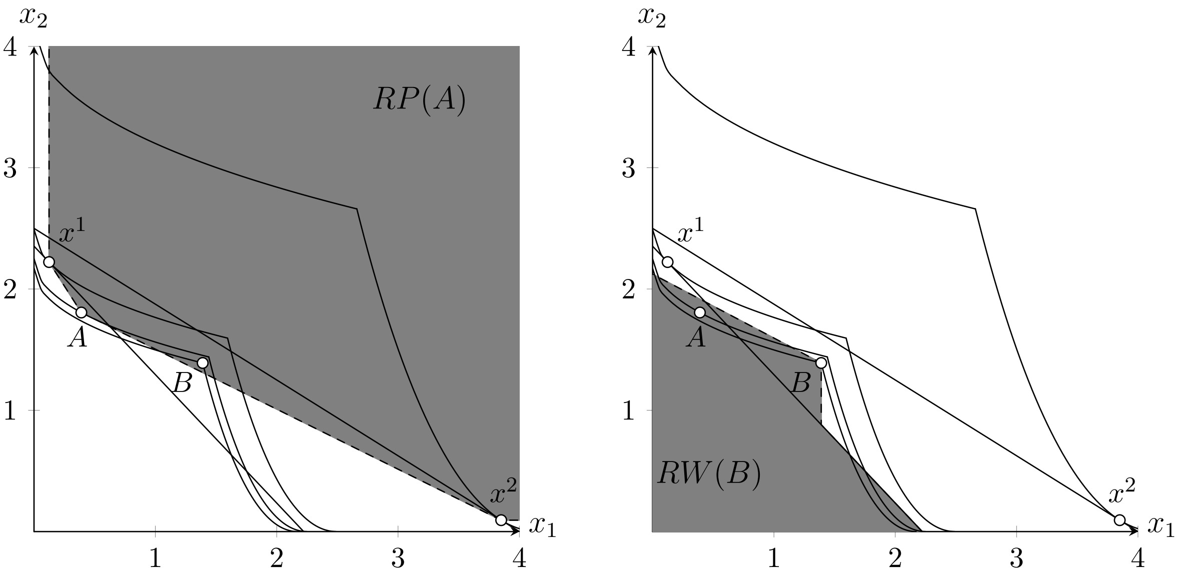
rank bundles differently. Consider, for example, the bundle $C = (44, 44)$ in Figure 3.1, which is ranked lower than the chosen bundles by the DM's preferences, but higher than the chosen bundles by the nicely behaved utility function.

Moreover, consider the revealed preferred set for the unobserved bundle $D = (26, 55)$ in Figure 3.1. As seen in the figure, $C \in RP(D)$. Indeed, by the nicely behaved function v, Bundle D is ranked below the two observed bundles while Bundle C is ranked above them. In fact, every nicely behaved (continuous, concave and monotone) utility function that rationalizes the DM's choices ranks Bundle D below Bundle C. However, by the DM's non-convex preferences, represented by u, Bundle C lies strictly below the indifference curve that goes through Bundle D, thus violating Varian's Fact 5. As a consequence, an outside observer (e.g. firm, researcher) who relies on Varian's method will reach a wrong conclusion when predicting a pairwise choice between bundles C and D.

3.2 Non-Expected Utility

Suppose a DM has to decide how to allocate a wealth of 1 between consumption in two mutually exclusive, exhaustive and equally probable states of the world. The allocation is attained by holding a portfolio of Arrow securities with unit prices $p = (p_1, p_2)$. Figure 3.2 presents a data set D of two observations. Portfolio $x^1 = (0.124, 2.222)$ is chosen when prices are $p^1 = (0.450, 0.425)$, and portfolio $x^2 = (3.850, 0.094)$ is chosen when prices are $p^2 = (0.250, 0.400)$. Notice that since $p^2 < p^1$, every portfolio that is feasible under p^1 is also feasible when prices are p^2, therefore $x^2 R_D^0 x^1$. Now consider two unobserved portfolios $A = (0.390, 1.806)$ and $B = (1.390, 1.390)$. Portfolio A is feasible under both prices, but portfolio B is feasible only under p^2. The revealed preferred set of A and the revealed worse set of B are drawn in panels 3.2a and 3.2b, respectively. Now consider the following utility function over portfolio $x = (x_1, x_2)$:

$$u(x_1, x_2) = \sqrt{\max\{x_1, x_2\}} + \frac{1}{4}\sqrt{\min\{x_1, x_2\}} \tag{3.2}$$



(a) Violation of the Revealed Preferred Set

(b) Violation of the Revealed Worse Set

Figure 3.2: Violations of Fact 5

which represents the preferences of an elation seeking DM (Gul, 1991) with $\beta = -0.75$ and a CRRA utility index with $\rho = 0.5$ over Arrow securities.[4] Therefore, the DM's preferences are not convex and $u(\cdot)$ is not quasi-concave (let alone not concave). The indifference curves drawn in Figure 3.2 through x^1 and x^2 demonstrate that this utility function rationalizes the data.

Figure 3.2a clearly demonstrates that while $B \in RP(A)$, it is not true that $B \in P_u(A)$. Similarly, Figure 3.2b shows that while $A \in P_u(B)$ it is not true that $A \in NRW(B)$. That is, the ranking of unobserved portfolios implied by the revealed preferred and revealed worse sets is inconsistent with the ranking of portfolios induced by a utility function that rationalizes the data. Again, an outside observer who relies on Varian's method will reach a wrong conclusion

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[4]

$$(x,y)^d(p_1,p_2) = \begin{cases} \left(\frac{p_2}{16p_1^2+p_1p_2}, \frac{16p_1}{16p_1p_2+p_2^2}\right) & if \quad \frac{p_1}{p_2} < 1 \\ \left\{\left(\frac{p_2}{16p_1^2+p_1p_2}, \frac{16p_1}{16p_1p_2+p_2^2}\right), \left(\frac{16p_1}{16p_1p_2+p_2^2}, \frac{p_2}{16p_1^2+p_1p_2}\right)\right\} & if \quad \frac{p_1}{p_2} = 1 \\ \left(\frac{16p_1}{16p_1p_2+p_2^2}, \frac{p_2}{16p_1^2+p_1p_2}\right) & if \quad \frac{p_1}{p_2} > 1 \end{cases}.$$

when predicting a pairwise choice between portfolios A and B.

4 Discussion

4.1 The Technical Issue

Both examples suggest the source of the above inconsistency with Varian's Fact 5. The failure of the non-parametric bounds can be traced back to the construction of the revealed preferred and revealed worse sets. Since by Afriat's Theorem if the data satisfies GARP there *exists* a concave utility function that rationalizes it, $S(x)$ (Definition 5) is non-empty for every x. However, there may exist a utility function that rationalizes the data for which there is no price vector p that supports x as an optimal choice. Therefore, even if x' is such that $xP_{D\cup\{p,x\}}x'$ for every $p \in S(x)$, it does not imply that a utility function that *never* chooses x ranks x above x'. In Figure 3.2a, for example, $BP_{D\cup\{p,B\}}A$ for every $p \in S(B)$, however the utility function that generated the DM's choices never chooses B and therefore may rank B below A.[5] Hence, Fact 5 fails since Varian's non-parametric bounds are constructed assuming that every bundle can be observed given some prices, while when the preferences are non-convex, some bundles are never chosen.

4.2 Implications

In many environments non-convex preferences are crucial and prevalent (e.g. risk, ambiguity and other-regarding preferences). The examples above demonstrate that when constructing non-parametric bounds through the method suggested in Varian (1982), the assumption of convexity of preferences is implicitly invoked. In particular, prediction and welfare analysis in contexts

[5]Definitions 5 and 6 can be trivially extended to include observed bundles, and then a similar argument can be constructed for the observed portfolio x^1 in Figure 3.2a. Note that the violation of the revealed worse set demonstrated in Figure 3.2b cannot occur for an observed bundle since there exists a price vector p that supports the bundle as an optimal choice. In fact, it is easy to show that $u(x)$ rationalizes D if and only if for every observed bundle x^i, $P_u(x^i) \subset NRW(x^i)$.

where non-convex preferences are frequently identified may suffer from the implementation of this method. In such cases, the convexification of the indifference curve may lead to a wrong prediction of behavior and therefore to an erroneous counterfactual analysis.

Both examples clearly demonstrate this issue. In these cases, every set of convex preferences that is consistent with the DM's choices, ranks some unsupportable (with respect to the true preferences) bundle higher than some other unobserved bundle. Therefore, a welfare or prediction analysis that is based on the non-parametric bounds may erroneously rank the two bundles, compared to the actual, non-convex, preferences held by the DM.

The identification of non-convex preferences becomes therefore a crucial step for non-parametric welfare analysis. Afriat theorem guarantees that such preferences cannot be identified by choices from linear budget lines. Therefore, more general menus (e.g. pairwise comparisons as in Halevy et al. (2016), see also Forges and Minelli (2009); Heufer (2012)) must be employed in order to identify the extent of potential non-convexities before proceeding to welfare analysis that is based on non-parametric bounds assuming the DM holds convex preferences.

An alternative approach may be to construct bounds using weaker assumptions on the true preferences. While these bounds would be looser, they will provide more reliable predictions and welfare analysis. In the following section we suggest one such alternative which is based only on the assumption of monotonicity of preferences.[6]

5 Alternative Bounds

5.1 Preliminaries

The preferences of a DM are considered (strictly) monotonic if every bundle is ranked (strictly) lower than all the bundles that include (strictly) greater

[6] Local non satiation is too weak an assumption to be used for the construction of bounds on the indifference curves since it provides information only on the existence of a better bundle, but not on its properties (e.g. direction).

quantities in each element.

Definition 7. A bundle $x \in \Re^K_+$ is

1. *monotonically preferred* to a bundle $y \in \Re^K_+$, denoted xMy, if $\forall i \in \{1, \ldots, K\} : x_i \geq y_i$.

2. *strictly monotonically preferred* to a bundle $y \in \Re^K_+$, denoted $xSMy$, if $\forall i \in \{1, \ldots, K\} : x_i > y_i$.

In the context of goods, an observer evaluates a bundle x to be better than another bundle y, either because x is observed as preferred to y, or x is monotonically preferred to y or a combination of these two through other bundles. The *monotonically revealed preference* relations formalize this idea.

Definition 8. Let $D = \left\{(p^i, x^i)_{i=1}^n\right\}$. A bundle $x \in \Re^K_+$ is[7]

1. *directly monotonically revealed preferred* to a bundle $y \in \Re^K_+$, denoted $xMR^0_D y$, if xMy or $xR^0_D y$.

2. *strictly directly monotonically revealed preferred* to a bundle $y \in \Re^K_+$, denoted $xSMP^0_D y$, if $xSMy$ or $xP^0_D y$.

3. *monotonically revealed preferred* to a bundle $y \in \Re^K_+$, denoted $xMR_D y$, if there exists a sequence of observed bundles $\left(x^j, x^k, \ldots, x^m\right)$ such that $xMR^0_D x^j, x^j R^0_D x^k, \ldots, x^m R^0_D y$.[8]

4. *strictly monotonically revealed preferred* to a bundle $y \in \Re^K_+$, denoted $xSMP_D y$, if there exists a sequence of observed bundles $\left(x^j, x^k, \ldots, x^m\right)$ such that $xMR^0_D x^j, x^j R^0_D x^k, \ldots, x^m R^0_D y$ and at least one of them is strict.

[7] Heufer (2012) and Korenok et al. (2013) define similar relations. Both go on to define an equivalent to GARP (M-GARP in Heufer (2012) and Monotonic Consistency in Korenok et al. (2013)). Heufer (2012) is interested in characterizing the equivalent to the revealed preferred set while Korenok et al. (2013) are concerned with the existence of a rationalizing utility function.

[8] Note that if x^i is an observed bundle then $x^i M z \implies x^i R^0_D z$, so MR_D is the transitive closure of MR^0_D.

Lemma 1. *Let* $D = \left\{(p^i, x^i)_{i=1}^n\right\}$ *be a data set of choices from linear budget lines that satisfies GARP. Let* $u(\cdot)$ *be a monotonic utility function that rationalizes the data. Then,* $xSMP_D y$ *implies* $u(x) > u(y)$*.*

Proof. See Appendix A. $\square$

5.2 An Alternative Fact 5

For every bundle x, we use the monotonically revealed preference relations to construct lower and upper bounds on the strictly upper contour set through x, using the following sets.

Definition 9. For every bundle $x \in \Re_+^K$:

1. The *monotonically revealed worse set* is $MRW(x) \equiv \{y \,|xSMP_D y\}$.

2. The *not monotonically revealed worse set,* denoted by $NMRW(x)$, is the complement of $MRW(x)$.

3. The *monotonically revealed preferred set* is $MRP(x) \equiv \{y \,|ySMP_D x\}$.

The equivalent to Varian (1982) Fact 5, using only the monotonicity of preferences assumption is

Proposition 1. *Let* $D = \left\{(p^i, x^i)_{i=1}^n\right\}$ *be a data set of choices from linear budget lines that satisfies GARP. Let* $u(\cdot)$ *be a monotonic utility function that rationalizes the data. Then for all bundles* x*,* $MRP(x) \subseteq P_u(x) \subseteq NMRW(x)$*.*[9]

Proof. Suppose $\hat{x} \in MRP(x)$. Then, $\hat{x}SMP_D x$. By Lemma 1 $u(\hat{x}) > u(x)$. Therefore, by Definition 4, $\hat{x} \in P_u(x)$. Hence, $MRP(x) \subseteq P_u(x)$.

Next, suppose $\hat{x} \in MRW(x)$. Therefore, $xSMP_D\hat{x}$. By Lemma 1 $u(x) > u(\hat{x})$. Therefore, by Definition 4, $\hat{x} \notin P_u(x)$. Hence, $P_u(x) \cap MRW(x) = \emptyset$. Thus, $P_u(x) \subseteq NMRW(x)$. $\square$

[9]Proposition 4.3 in Heufer (2012) implies that $MRP(x)$ is the tightest inner bound for the strictly upper contour set through the bundle x.

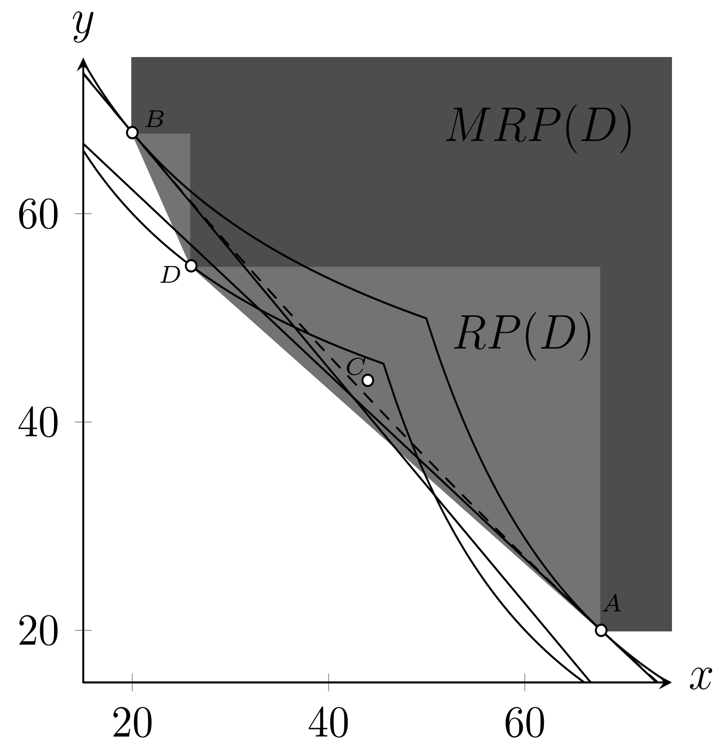


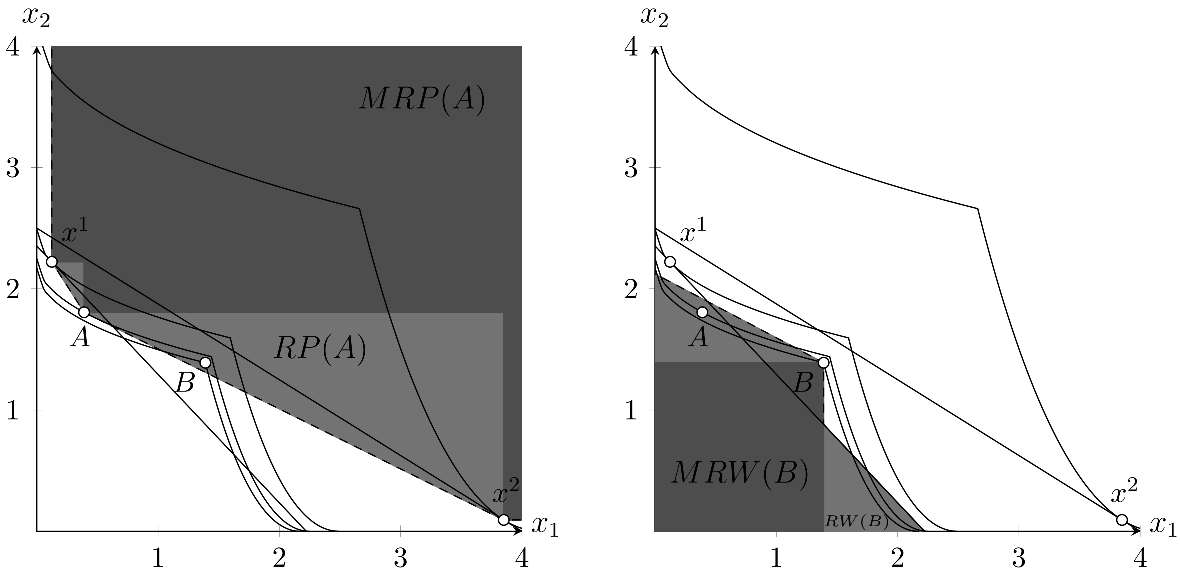
Figure 5.1: Textbook example revisited

5.3 The Examples Revisited

5.3.1 Textbook Example

In section 3.1 we considered a DM with a utility function described in (3.1). We showed in Figure 3.1 that this DM prefers Bundle D over Bundle C, although Bundle C was included in the revealed preferred set of Bundle D. We claimed that this discrepancy results from the convexity of preferences implicitly invoked by the construction suggested by Varian (1982).

Figure 5.1 demonstrates that by basing the construction of the non-parametric bounds solely on the monotonicity of preferences assumption, while dropping the convexity of preferences assumption, one may avoid such discrepancies. The dark gray area in Figure 5.1 designates the Monotonically Revealed Preferred set while the light gray shows the original Revealed Pre-



(a) Revealed Preferred Set revisited

(b) Revealed Worse Set revisited

Figure 5.2: Fact 5 revisited

ferred set. It is clear that using the alternative construction Bundle C no longer belongs to the set of bundles that are classified as preferred to D. Practically, this implies that in case of a pairwise choice between bundles C and D, an observer would no longer predict Bundle C to be chosen over Bundle D.[10]

5.3.2 Non-Expected Utility

In Section 3.2 we described an elation seeking DM that allocates her wealth between consumption in two mutually exclusive, exhaustive and equally probable states of the world. Figure 3.2 demonstrated that using Varian (1982) Fact 5, the safe Bundle B was included in the revealed preferred set constructed for the risky Bundle A (and Bundle A was a member of the revealed worse set

[10]If x is an unobserved bundle and there exists at least one observed bundle x^i that is directly revealed preferred to x but does not monotonically dominate x, then there are bundles that will be ranked above x using the convexity bound but are incomparable to x using the monotone bounds.

constructed for Bundle B).

Figure 5.2 depicts the monotonically revealed preferred set of Bundle A and the monotonically revealed worse set of Bundle B using the alternative bounds that assume only the monotonicity of preferences (again, the dark gray area designates the alternative sets while the light gray shows the original sets). These bounds suggest that the observed choices do not provide enough information to separate bundles A and B. In fact, there is not enough information to compare the risky Bundle A with any safe bundle that is not monotonically better. Hence, an observer using this alternative cannot rule out an elation seeking behavior that induces a preference for Bundle A over seemingly attractive safe bundles.

6 Conclusions

In this short paper we draw attention to the assumption of convexity of preferences implicitly invoked in the construction of non-parametric bounds on indifference curves as suggested by Varian (1982). We then suggest a similar construction that refrains from using the assumption of convexity of preferences and is based solely on the premise that in the context of goods, an observer evaluates a bundle x to be better than another bundle y, either because x is observed to be preferred to y, or x monotonically dominates y or a combination of these two.

As demonstrated in figures 5.1 and 5.2 the assumption of monotonicity of preferences is also included implicitly in the original construction. Therefore the alternative construction provides revealed preferred and revealed worse sets that are subsets of the original sets. Hence, the price of the more reliable bounds obtained by dropping the assumption of convexity of preferences is less predictive power and weaker ability to provide conclusive welfare analysis.

A Proof of Lemma 1

Definition 10. A utility function $u : \Re_+^K \rightarrow \Re$ is

1. Locally non satiated if $\forall x \in \Re_+^K$ and $\forall \epsilon > 0$, $\exists y \in B_\epsilon(x) \cap \Re_+^K$ such that $u(x) < u(y)$.

2. Monotone if xMy implies $u(x) \geq u(y)$ and $xSMy$ implies $u(x) > u(y)$.

Lemma 2. *If $u(\cdot)$ is a locally non satiated utility function that rationalizes $D = \left\{ (p^i, x^i)_{i=1}^n \right\}$, then $x^i P_D^0 x$ implies $u(x^i) > u(x)$.*

Proof. If $x^i P_D^0 x$ then $x^i R_D^0 x$. Since $u(\cdot)$ rationalizes D, by Definition 3, $x^i R_D^0 x$ implies $u(x^i) \geq u(x)$. Suppose that $u(x^i) = u(x)$. Since $p^i x^i > p^i x$, $\exists \epsilon > 0$ such that $\forall y \in B_\epsilon(x) : p^i x^i > p^i y$. By local non satiation $\exists y' \in B_\epsilon(x)$ such that $u(y') > u(x) = u(x^i)$. Thus, y' is a bundle such that $p^i x^i > p^i y'$ and $u(y') > u(x^i)$, in contradiction to $u(\cdot)$ rationalizing D. Therefore, $u(x^i) > u(x)$. $\square$

Now, we are ready to prove Lemma 1: **Let $\mathbf{D} = \left\{ (\mathbf{p^i}, \mathbf{x^i})_{\mathbf{i=1}}^{\mathbf{n}} \right\}$ be a data set of choices from linear budget lines that satisfies GARP. Let $\mathbf{u}(\cdot)$ be a monotonic utility function that rationalizes the data. Then, $\mathbf{xSMP_Dy}$ implies $\mathbf{u(x) > u(y)}$.**

Proof. Suppose $xSMP_Dy$. Hence, by Definition 8.4, there exists a sequence of observed bundles $\left(x^j, x^k, \ldots, x^m\right)$ such that $xMR_D^0x^j, x^jR_D^0x^k, \ldots, x^mR_D^0y$ and at least one of them is strict.

If $xMR_D^0x^j$ is strict then $xSMP_D^0x^j$, that is: $xSMy$ or xP_D^0y. Since $u(\cdot)$ is monotone and rationalizes D then by Definition 10.2 and Lemma 2 $u(x) > u(x^j)$. In addition, since $u(\cdot)$ rationalizes D, $u(x^j) \geq u(x^k), \ldots, u(x^m) \geq u(y)$. Thus, there exists a sequence of observed bundles $\left(x^j, x^k, \ldots, x^m\right)$ such that $u(x) > u(x^j), u(x^j) \geq u(x^k), \ldots, u(x^m) \geq u(y)$. Therefore, $u(x) > u(y)$.

Otherwise, $xMR_D^0x^j$ implies xMy or xR_D^0y. Since $u(\cdot)$ is monotone and rationalizes D then $u(x) \geq u(x^j)$. But then at least one of $x^jR_D^0x^k, \ldots, x^mR_D^0y$ is strict. Thus, by Lemma 2, $u(x^j) \geq u(x^k), \ldots, u(x^m) \geq u(y)$ such that at least one of the inequalities is strict, which implies that $u(x) > u(y)$. $\square$

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