ALLAIS MEETS STROTZ: REMARKS ON THE RELATION BETWEEN PRESENT BIAS AND THE CERTAINTY EFFECT

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ABSTRACT. The paper establishes a tight relation between non-standard behaviors in the domains of risk and time by considering a decision maker with non-expected utility preferences who believes that only present consumption is certain while any future consumption is uncertain. We provide the first complete characterization of the two-way relations between i) certainty effect and present bias, and, ii) common ratio effect and the common difference effect. A corollary to our results is that hyperbolic discounting implies the Common Ratio Effect and that quasi-hyperbolic discounting implies the Certainty Effect.

1. Introduction

Almost all decisions involve outcomes that are uncertain, realized at different points in time, or both. For example, following a strict and often unpleasant diet program requires some motivation about future gains accruing from it, which are quite often uncertain. There has been persistent interest in the fields of Psychology and Economics to understand how behaviors across risky and temporal domains might be related to each other. The standard approach of modeling intertemporal preferences is through the use of geometric (constant, exponential) discounting in which the payoff of a stream of consumption is aggregated through a (delay-geometric) weighting that results in a present discounted value. This is mirrored in the risk domain, as the canonical model for risk behavior is expected utility which aggregates the utility of each possible alternative by weighting it by its probability. But the similarities do not end here as both models contain similar inadequacies as descriptive models. First, preferences are disproportionately sensitive to certainty (certainty

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effect) and to the present (present bias/immediacy effect/diminishing impatience). Second, proportional changes in probabilities or equal changes in time delays for timed consumption affect preferences disproportionately (common ratio effect and common difference effect respectively).[1] This two-way relation is well accepted in the Psychological literature (Keren and Roelofsma, 1995; Green and Myerson, 2004; Weber and Chapman, 2005; Chapman and Weber, 2006, to name a few) and there is an understanding that the existence of such mirroring behaviors is not a mere coincidence, but points to a common fundamental property of decision making that manifests itself in different domains of behavior (Prelec and Loewenstein, 1991). There are many ways in which this relation between risk and temporal behavior can be motivated. Delayed rewards or consumption can be inherently risky, as there might be events between the current date and the promised date which interfere in the process of acquiring the reward/consumption. On the other hand, Rachlin et al. (1986; 2000) suggested that the certain value of probabilistic rewards may be expressed not directly by probabilities but by mean waiting time, and the form of the waiting-time discount function is similar to that used in a model of temporal behavior consistent with present bias. This relation has also been analyzed in more recent works in Economics (Halevy, 2008; Saito, 2011; Baucells and Heukamp, 2012; Epper and Fehr-Duda, 2015; Saito, 2015). Given this is a two way relation, none of risk or temporal behaviors have primacy over the other, so any formalization of this relation would necessarily involve the two-way feature discussed above. The goal of this paper is to provide a formal characterization of this relation in the most natural setting. We start by showing how previous attempts at this endeavor have failed to achieve this goal. To be more specific, we show that though the formalization in the direction from certainty effect to diminishing impatience has been correctly posited in the literature, it is the converse relation that still lacks formal rigor. We provide a formal characterization of the two-way relations between i) certainty effect and present bias, and, ii) common ratio effect and the common difference effect. A corollary to our results is that hyperbolic discounting implies

[1] Often times certainty effect and present bias are taken as special cases of common ratio effect and common difference effect, respectively.

the Common Ratio Effect and that quasi-hyperbolic discounting implies the Certainty Effect.

The next section provides a brief acknowledgment to the prior unsuccessful attempts made in this literature to establish risk-time equivalence relations. In Section 3 we suggest an intuitive extension to the existing notion of diminishing impatience, which when used in the analytical framework provided by Halevy (2008), re-establishes the severed connection between non-standard behavior over time and under risk.

2. Background

The idea that diminishing impatience (hyperbolic discounting, present bias) may be related to the certainty of the present and the risk associated with future rewards, was formalized by Halevy (2008). In this model, every consumption path $\mathbf{c} = (c_0, c_1, c_2, \ldots)$ is subject to a constant hazard rate of termination (r). The decision maker chooses among consumption paths as if she calculates present discounted utility for every possible length of the path (periods before stopping). The distribution over present discounted utilities is then evaluated using Rank Dependent Utility (RDU) with probability weighting function $g(\cdot)$, which models preferences that are disproportionately sensitive to certainty. The crucial behavioral axiom accommodates dynamic inconsistency between optimal choices at the present and the immediate future $(t = 1)$ only if there is uncertainty concerning consumption in the immediate future, drawing an important qualitative distinction between the effect of randomness in the immediate future and stochastic consumption in later periods $(t = 2, 3, \ldots)$.[2] Together with other standard axioms on the DM's preferences over stochastic consumption streams, they are then represented by the utility function:

$$U(\mathbf{c}, r) = \sum_{t=0}^{\infty} g\left((1-r)^t\right) \delta^t u(c_t) \tag{1}$$

where δ is a constant pure time preference parameter and $u(\cdot)$ is her felicity function. The decision maker's impatience at time t is then the ratio of her discount function at periods t and $t+1$. Halevy (2008) defines diminishing

[2] Which is impossible to draw in a framework in which consumption occurs only in a single period.

impatience if the impatience is maximized at $t = 0$, and Theorem 1 in his paper claims equivalence between diminishing impatience and increasing elasticity of $g(\cdot)$. To prove his claim, Halevy (2008) proceeds in two steps. First, diminishing impatience holds if and only if the weighting function satisfies a certain functional inequality.[3] Second, he invokes an equivalence result from Segal (1987, Lemma 4.1) between the above functional inequality and increasing elasticity of $g(\cdot)$. Saito (2011) correctly points out that Segal did not prove that increasing elasticity of the weighting function is necessary for the functional inequality, and provides an example of a DM who exhibits diminishing impatience but her weighting function's elasticity is not increasing (and therefore does not exhibit the common ratio effect).[4] Saito (2011) attempts to establish the sought equivalence between present bias and the certainty effect (Claim 3 in his paper) by retaining the first part of Halevy's argument, and noting that the functional inequality is equivalent (by definition) to the certainty effect for RDU.

We show that diminishing impatience as defined by Halevy (2008) and used by Saito (2011) does *not* imply the certainty effect. In light of this new finding, the equivalence results of Halevy (2008) and Saito (2011) reduce to a one-directional implication from the domain of risk to the domain of time. We provide details in Section 3.2.

3. Definitions and Results

3.1. The Certainty and Common Ratio Effects.

Let (x, p) be a lottery that pays x with probability $0 \leq p \leq 1$ and 0 with probability $1 - p$. A DM exhibits *Strict Certainty Effect* if for every $x, y \in \mathbb{R}_+$ and $p, q \in (0, 1)$: $(x, 1) \sim (y, q) \Rightarrow (x, p) \prec (y, pq)$. She exhibits *Certainty Effect* if for every $x, y \in \mathbb{R}_+$ and $p, q \in (0, 1)$: $(x, 1) \sim (y, q) \Rightarrow (x, p) \preceq (y, pq)$ and there exist p, q for which the preference is strict. If the DM's preferences are represented

[3] The functional inequality is a special case of Kahneman and Tversky (1979, pg. 282) subproportionality which characterizes common-ratio violations for RDU. Kahneman and Tversky also state the equivalence claimed later by Segal (1987, Lemma 4.1), which is used in the second part of Halevy's argument.

[4] In particular, Saito (2011) shows that Tversky and Kahneman (1992) weighting function for gains with $\gamma = 0.5$ exhibits diminishing impatience but does not possess increasing elasticity around 0 and does not satisfy the common ratio effect.

by RDU then Strict Certainty Effect is equivalent to the following restriction on the weighting function:[5]

$$g\left(pq\right) > g\left(p\right) g\left(q\right) \tag{2}$$

A DM exhibits *Strict Common Ratio Effect* if for every $x, y \in \mathbb{R}_+$ and $p, q \in (0,1)$, $\ell \in (0,1]$: $(x,\ell) \sim (y, q\ell) \Rightarrow (x, p\ell) \prec (y, pq\ell)$. She exhibits *Common Ratio Effect* if the implied preference is weak and there exist p, q, ℓ for which the preference is strict. If the DM's preferences are represented by RDU then Strict Common Ratio Effect is equivalent to the following restriction on the weighting function:[6]

$$\frac{g\left(\ell\right)}{g\left(p\ell\right)} > \frac{g\left(q\ell\right)}{g\left(pq\ell\right)} \tag{3}$$

3.2. **Diminishing Impatience.** We assume that the DM's preferences over stochastic consumption paths satisfy the behavioral axioms in Halevy (2008) and are represented by (1). The discount function at period t is: $\Delta\left(t\right) = \beta^t g\left(\left(1-r\right)^t\right)$ and her (one period) impatience at t is $\Delta\left(t\right)/\Delta\left(t+1\right)$. In Halevy (2008) and Saito (2011), the definition of Diminishing Impatience (DI) is restricted to only one-period delay. It implies that for all natural numbers t: $\Delta\left(0\right)/\Delta\left(1\right) > \Delta\left(t\right)/\Delta\left(t+1\right)$ which is satisfied if and only if for every $r \in (0,1)$ and $t \in \mathbb{N}$:[7]

$$g\left(\left(1-r\right)^{t+1}\right) > g\left(1-r\right) g\left(\left(1-r\right)^t\right) \tag{4}$$

Both Halevy (2008) and Saito (2011) state without proof that (2) holds if and only if (4) holds. Although the direction (2)→(4) is immediate,[8] we provide in Appendix A a counter-example which shows that the converse in not true in general. In other words, DI as defined above does *not* imply the certainty effect for arbitrary weighting functions. Intuitively, the certainty effect implies a bias towards certainty irrespective of how risky the alternative is, the dual to which would be a bias towards the present ($t = 0$) irrespective of the delay in the

[5] *Certainty Effect* implies weak inequality in (2) for every $x, y \in \mathbb{R}_+$ and $p, q \in (0,1)$ and existence of p, q for which (2) is satisfied with strict inequality.

[6] *Common Ratio Effect* implies weak inequality in (3) and existence of p, q, ℓ for which the inequality in (3) is strict.

[7] Note that this is equivalent to writing $g\left(r^{t+1}\right) > g\left(r\right) g\left(r^t\right)$ $\forall r \in (0,1)$ and $t \in \mathbb{N}$.

[8] Define $p := 1 - r$ and $q := \left(1-r\right)^t$

compared consumption. In evaluating the reason for the severed connection between time and risk preferences, we note that the definition of diminishing impatience used in the literature focuses on a delay of a single period, thus only comparing $\Delta(t)$ to $\Delta(t+1)$ as t increases from 0. This one-period definition fails to generalize to longer delays, and thus fails to account for present bias behaviorally.[9]

Motivated by the behavioral literature in general, and the quasi-hyperbolic discounting model in particular, which focus on the failure of stationarity independently of the delay under consideration,[10] we suggest to compare $\Delta(t)$ to $\Delta(t+k)$ for arbitrary $k \geq 1$.

Definition 1. The decision maker exhibits *Delay Independent Diminishing Impatience (DIDI)* if $\frac{\Delta(0)}{\Delta(k)} > \frac{\Delta(t)}{\Delta(t+k)}$ $\forall k, t \in \mathbb{N}$, where $\Delta(t)$ is the decision maker's time discounting at period t.

DIDI requires impatience to diminish for all possible delays ($k \geq 1$), hence is a strengthening of the standard definition,[11] which is satisfied by the quasi hyperbolic discounting model (see the Proposition below). For preferences represented by (1) DIDI holds if and only if for every $r \in (0,1)$ and $t, k \in \mathbb{N}$: $g\left((1-r)^{t+k}\right) > g\left((1-r)^{k}\right) g\left((1-r)^{t}\right)$.

Hyperbolic discounting motivates the definition of Strongly Diminishing Impatience as $\frac{\Delta(t)}{\Delta(t+1)} > \frac{\Delta(t')}{\Delta(t'+1)}$ $\forall t' > t \in \mathbb{N}$. Note that Strongly Diminishing Impatience too, is restricted to only one-period delays, and hence similar to Definition 1, we strengthen this measure to be delay independent:

Definition 2. The decision maker exhibits *Delay Independent Strongly Diminishing Impatience (DISDI)* if $\frac{\Delta(t)}{\Delta(t+k)} > \frac{\Delta(t')}{\Delta(t'+k)}$ $\forall k, t' > t \in \mathbb{N}$, where $\Delta(t)$ is the decision maker's time discounting at period t.

[9]For further discussion and intuition see the introductory discussion in Appendix A.
[10]Halevy (2015) provides a formal definition and recent experimental evidence for stationarity in a dynamic setting.
[11]DI is the special case of DIDI where delay $k = 1$. An implication of the counter-example provided in Appendix A is that DI does not imply DIDI.

If preferences are represented by (1) then DISDI holds if and only if for every $r \in (0,1)$ and $t < t',\ k \in \mathbb{N}\backslash\{0\}$: $\dfrac{g\left((1-r)^t\right)}{g\left((1-r)^{t+k}\right)} > \dfrac{g\left((1-r)^{t'}\right)}{g\left((1-r)^{t'+k}\right)}$.

Proposition. *Quasi-hyperbolic discounting satisfies DIDI (but not DISDI), Hyperbolic discounting satisfies DISDI (and hence DIDI).*

Proof. In case of quasi-hyperbolic discounting: $U = u(c_0) + \beta \sum_{t=1}^{\infty} \delta^t u(c_t)$, and for $\beta < 1$:

$$\frac{\Delta(0)}{\Delta(k)} = \frac{1}{\beta\delta^k} > \frac{\beta\delta^t}{\beta\delta^{t+k}} = \frac{1}{\delta^k} = \frac{\Delta(t)}{\Delta(t+k)} = \frac{\Delta(t')}{\Delta(t'+k)}$$

The last equality holds for $t, t' > 0$. Hence, quasi-hyperbolic discounting satisfies DIDI, but not DISDI.

In Hyperbolic Discounting the discount function for period t is given by $\Delta(t) = \dfrac{1}{1+\rho t}$ for $\rho > 0$. For arbitrary k, and $t' > t$,

$$\frac{\Delta(t)}{\Delta(t+k)} = 1 + \frac{\rho k}{1+\rho t} > 1 + \frac{\rho k}{1+\rho t'} = \frac{\Delta(t')}{\Delta(t'+k)}$$

Hence, hyperbolic discounting satisfies DISDI (and hence DIDI). □

3.3. **The Relation between Risk and Time Preferences.** As noted above, the effect of risk attitude on intertemporal preferences in (1) is straightforward. We summarize this relation below.

Claim. Strict Certainty Effect (2) implies Delay Independent Diminishing Impatience (DIDI), and the Strict Common Ratio Effect (3) implies Delay Independent Strongly Diminishing Impatience (DISDI).

The following Theorem proves the converse direction (though in a weaker form that does not substantiate an equivalence), that is - how the DM's intertemporal preferences determine her risk attitudes.[12] The result is direct and

[12]Note that although the Theorem does not imply *Strict* Common Ratio/Certainty Effects, it is *inconsistent* with expected utility since even the weaker forms imply the existence of probabilities for which (2) and (3) are satisfied with strict inequality.

comprehensive in the sense that it does not rely on any intermediate connections through properties (e.g, convexity, increasing elasticity) of the weighting function.

Theorem. *Consider a DM represented by (1) with continuous $g(\cdot)$.*

(1) *Delay Independent Strongly Diminishing Impatience implies the Common Ratio Effect (and the Certainty Effect).*
(2) *Delay Independent Diminishing Impatience implies the Certainty Effect.*

Proof. (1) Consider a sequence $\{\frac{m_i}{n_i}\}_{i=1}^{\infty}$ of rational numbers that converges to $\frac{ln\,p}{ln\,q\ell}$, where m_i, n_i are positive integers. Similarly, consider a sequence $\{\frac{a_i}{b_i}\}_{i=1}^{\infty}$ of rational numbers that converges to $\frac{ln\,\ell}{ln\,q\ell}$, where a_i, b_i are positive integers. Note that $\frac{ln\,\ell}{ln\,q\ell} < 1$, so we can choose $\{\frac{a_i}{b_i}\}_{i=1}^{\infty}$ such that $a_i < b_i$. Now, given this sequences, define a sequence $\{r_i\}$, such that $q\ell = r_i^{n_i b_i}$, that is $r_i = (q\ell)^{\frac{1}{n_i b_i}} < 1$. Note that as $\frac{a_i}{b_i}$ converges to $\frac{ln\,\ell}{ln\,q\ell}$, $r_i^{a_i n_i} = (q\ell)^{\frac{a_i}{b_i}}$ converges to $(q\ell)^{\frac{ln\,\ell}{ln\,q\ell}} = \ell$. Similarly, as $\frac{m_i}{n_i}$ converges to $\frac{ln\,p}{ln\,q\ell}$, $r_i^{m_i b_i} = (q\ell)^{\frac{m_i}{n_i}}$ converges to $(q\ell)^{\frac{ln\,p}{ln\,q\ell}} = p$.

Now using DISDI, $\forall i$:

$$\frac{g\left(r_i^{a_i n_i}\right)}{g\left(r_i^{a_i n_i + m_i b_i}\right)} > \frac{g\left(r_i^{n_i b_i}\right)}{g\left(r_i^{n_i b_i + m_i b_i}\right)}$$

Using the continuity of g, as $i \to \infty$, the Common Ratio Effect follows:

$$\frac{g(\ell)}{g(p\ell)} \geq \frac{g(q\ell)}{g(pq\ell)}$$

(2) Let $p, q \in (0,1)$ and assume without loss of generality that $p < q$. Consider a sequence $\{\frac{m_i}{n_i}\}_{i=1}^{\infty}$ of rational numbers that converges to $\frac{ln\,p}{ln\,q}$, where m_i, n_i are positive integers. Given this sequence, define a sequence $\{r_i\}$, such that $q = r_i^{n_i}$, that is: $r_i = q^{\frac{1}{n_i}} < 1$. Note that as $\frac{m_i}{n_i}$ converges to $\frac{ln\,p}{ln\,q}$, $r_i^{m_i} = q^{\frac{m_i}{n_i}}$ converges to $q^{\frac{ln\,p}{ln\,q}} = p$.

Now, $\forall i$:

$$\begin{aligned} g(r_i^{m_i+n_i}) &> g(r_i^{m_i})g(r_i^{n_i}) \\ g(r_i^{m_i}q) &> g(r_i^{m_i})g(q) \end{aligned}$$

Using the continuity of g, Certainty Effect follows: $g(pq) \geqslant g(p)g(q)$. □

Corollary. *Consider a DM represented by (1) with continuous $g(\cdot)$.*

(1) *Hyperbolic discounting implies the Common Ratio Effect (and the Certainty Effect).*
(2) *Quasi-hyperbolic discounting implies Certainty Effect.*

It is important to recall that preferences represented by (1) are defined over consumption streams in discrete time (following Koopmans, 1960).[13] It follows that all notions of diminishing impatience (as DI, DIDI, DISDI) are required to hold only for natural numbers, while risk preferences (*Certainty Effect*, *Common Ratio Effect*) are defined over lotteries with probabilities in the simplex. With this insight, it is not surprising that properties of risk preferences manifest themselves in the time domain. The counter-example in the Appendix together with the Theorem demonstrate that the opposite direction can be established as well, but the notion of diminishing impatience must be appropriately defined so it will not be delay dependent. We believe that these new notions (DIDI and DISDI) are very intuitive and reflect the natural meaning of diminishing impatience. Moreover, in light of recent work generalizing hyperbolic discounting to continuous time (Webb, 2016) we conjecture that continuous adaptations of DIDI and DISDI will be required in order to create the link from time to risk, though this remains for future work as the behavioral underpinning of (1) are stated in discrete time.

Appendix A. Diminishing Impatience does not imply the Certainty Effect

We start by providing a basic intuition of why DI as characterized by (4) fails to imply the certainty effect. To complete a proof in the direction from DI to certainty effect one is required to approximate arbitrary probabilities used in lotteries by the total hazard rate of termination over one or multiple periods. More specifically, one needs to approximate the ratio of probabilities

[13] This framework is considerably different from Fishburn and Rubinstein (1982) whose domain includes payments of $\$x$ at time t, which is applicable to more selective environments (as bargaining).

$\frac{g(p)}{g(pq)}, \frac{g(1)}{g(q)}$ in the relation (2) by the relative hazard rates between two consecutive time-periods in (4), $\frac{g((1-r)^t)}{g((1-r)^{t+1})}, \frac{g(1)}{g(1-r)}$ respectively, for some hazard rate r. Given (4), we are restricted to establishing the certainty effect relation for p, q combinations which can be approximated as integer exponents of each other, hence the result does not generalize to the certainty effect. Under DISDI we are approximating $\frac{g(p)}{g(pq)}, \frac{g(1)}{g(q)}$ by the relative hazard rates between arbitrarily spaced time-periods in (4), $\frac{g((1-r)^t)}{g((1-r)^{t+k})}, \frac{g(1)}{g((1-r)^k)}$. Hence, we are allowed to approximate p, q by different integer exponents of the hazard rate and hence rational exponents of each other (for example, when, $p = r^k, q = r^t$, then $p = q^{\frac{k}{t}}$). Given the rationals are dense in reals (and the integers are not!), a sequence of $\frac{k}{t}$'s can approximate $ln_q p$ and this allows the relation from time to risk be established for general p, q and continuous g. The following counter-example provides the vital step that DI does not imply DISDI.

If (4) implied (2), then (4) would also imply that $\forall r \in (0,1)$ and $\forall m, n \in \mathbb{N}$

$$g(r^{m+n}) > g(r^m)g(r^n) \tag{5}$$

We rewrite this problem in an additive form by defining $f(x) = -log(g(e^{-x})) \iff g(x) = e^{-f(-logx)}$. Then $f : (0,\infty) \to (0,\infty)$ is differentiable and increasing, just like the function g. The inequalities under consideration are now:

$$\begin{aligned}
\forall t \in \mathbb{N} \text{ and } \forall r \in (0,1),\ g(r^{t+1}) &> g(r)g(r^t) \\
\iff e^{-f\left(-log(r^{t+1})\right)} &> e^{-f\left(-log(r^t)\right)}e^{-f(-log(r))} \\
\iff f(-(t+1)log(r)) &< f(-tlog(r)) + f(-log(r))
\end{aligned}$$

Now, defining $x := -log(r) \in (0,\infty)$ for $r \in (0,1)$.

$$f((t+1)x) < f(tx) + f(x) \tag{6}$$

Further, the boundary conditions $g(0) = 0$ and $g(1) = 1$ translate to $f(0) = 0$ and $f(\infty) = \infty$.[14]

[14] Using the extended real line $(\mathbb{R} \cup \infty)$

Similarly, (5) converts to

$$f(mx+nx) < f(mx)+f(nx) \quad \forall x \in (0,\infty) \, and \, \forall m,n \in \mathbb{N} \tag{7}$$

Summing it up, (4) implies (5), if and only if (6) implies (7). The next step is to propose a function f which would satisfy (6) on all points of its domain, but violate (7) for some $x \in \mathbb{R}$ and some $m, n \in \mathbb{N}$.

Instead of providing the function f, we propose it's derivative h, so f can be calculated as $f(x) = \int_0^x h(x)dx$.[15] Let, $k = \frac{20}{1+\sin(\pi/2-.0001)}$ and $\delta = 50k\pi\cos(\pi/2-.0001) \approx .157$.

Let,

$$h(x) = \begin{cases} 11+(1-x)\delta & For\ x<1 \\ 1+\frac{k}{2}+\frac{k}{2}\sin 100\pi(1+\frac{\pi/2-.0001}{100\pi}-x) & For\ 1 \le x \le 1.005+\frac{\pi/2-.0001}{100\pi} \\ 1 & For\ 1.005+\frac{\pi/2-.0001}{100\pi} < x < 2-.005 \\ 4+3\sin 100\pi(x-2) & For\ 2-.005 \le x \le 2+.005 \\ 7 & For\ 2+.005 < x < 2.5-.005 \\ 4+3\sin 100\pi(2.5-x) & For\ 2.5-.005 \le x \le 2.5+.005 \\ 1 & For\ 2.5+.005 < x < 3-.005 \\ 4+3\sin 100\pi(x-3) & For\ 3-.005 \le x \le 3+.005 \\ 7 & For\ 3+.005 < x < 5-.005 \\ 4+3\sin 100\pi(5-x) & For\ 5-.005 \le x \le 5+.005 \\ 1 & For\ x > 5+.005 \end{cases}$$

f is increasing, twice differentiable and $f(\infty) = \infty$. $h(x)$ is plotted in Figure 1.

We next show that (6) holds.

Lemma 3. $\forall t \in \mathbb{N},\ \forall x \in \mathbb{R}, \int_0^x h(x)dx > \int_{tx}^{(t+1)x} h(x)dx.$

Proof. The most intuitive way to check the claim would be to notice that the sinusoids introduced hardly perturb the area under the curve. Figure 2 illustrates the point in a more clear fashion by considering the function h for

[15]Recall that $f(0) = 0$.

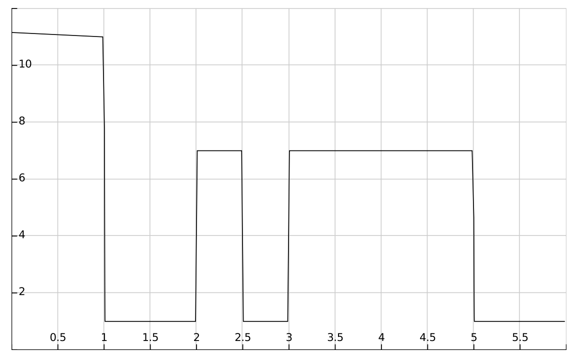

FIGURE 1. The function h.

a small part of the real line. For all practical purposes, one could go about checking the inequalities by replacing the sinusoid (in black) in Figure 1 by a corresponding discontinuous function($\bar{h}(x) = 7$ for $x \leq 2.5$, $\bar{h}(x) = 1$ for $x > 2.5$ as drawn in red). The area between the two curves in $[2.495, 2.5]$ is only $(.005 * 3 - \frac{3}{100\pi}) \approx .005$. Therefore, as long as the inequalities hold with a large enough margin, this intuitive method of approximating sinusoids with flat lines works fine. The area between the two curves in $[2.5, 2.505]$ is also $(.005 * 3 - \frac{3}{100\pi})$. Thus, the two approximations in $[2.495, 2.505]$ are equal and opposite in direction, and the areas under the red and black curves in this region are equal. During our analysis, in some cases there will be multiple approximations in opposite directions which would partially or completely cancel each other out.

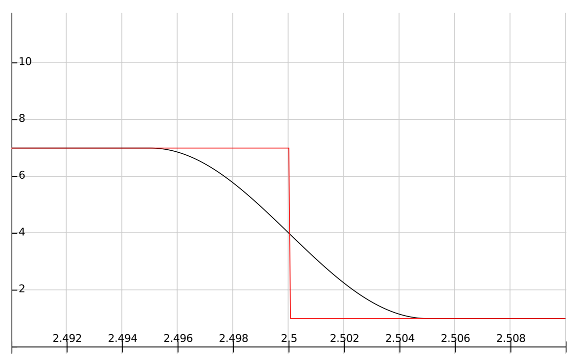

FIGURE 2. Function h approximated in a sinusoidal region

Utilizing this intuition more rigorously, one can create upper bounds and lower bounds on $\int_{tx}^{(t+1)x} h(x)dx$ and $\int_0^x h(x)dx$ respectively to complete the proof.
For $0 < x \leq 1$, $\int_0^x h(x)dx > \int_{tx}^{(t+1)x} h(x)dx$ is obvious, as $[0, x]$ contains the highest values obtained by $h(x)$ on the real line.
For, $1 < x \leq \frac{5}{3}$, $\int_0^x h(x)dx = \int_0^1 h(x)dx + \int_1^x h(x)dx > \frac{1}{2}(11+11+\delta)+(x-1) = 10 + \frac{\delta}{2} + x$.[16] The inequality holds because $h(x) \geq 1$ with strict inequality for $1 \leq x < 1.005 + \frac{\pi/2-.0001}{100\pi}$, and hence $\int_1^x h(x)dx > x - 1$. In the interval $[tx, (t+1)x]$, $h(x) \leq 7$ and after mutual canceling out there are no more than 3 sinusoidal perturbations which could increase the area under the curve. Hence, $\int_{tx}^{(t+1)x} h(x)dx < 7x + 3(.015 - \frac{3}{100\pi}) = 6x + x + 3(.015 - \frac{3}{100\pi}) \leq 6(\frac{5}{3}) + x + 3(.015 - \frac{3}{100\pi}) = 10 + x + 3(.015 - \frac{3}{100\pi})$.
For $\frac{5}{3} \leq x \leq 2$, $\int_0^x h(x)dx > 10 + \frac{\delta}{2} + x$ as before. On the other hand, using the same line of logic as before, $\int_x^{2x} g(x)dx < 1.x + 6[(4-3) + (2.5 - 2)] + 3.(.015 - \frac{3}{100\pi}) = 9 + x + 3.(.015 - \frac{3}{100\pi})$. Similarly, $\int_{2x}^{3x} h(x)dx \leq 1.x + 6[5 - 2.\frac{5}{3}] + 3.(.015 - \frac{3}{100\pi}) = 10 + x + 3.(.015 - \frac{3}{100\pi})$.
Similarly for larger values of x, it can be shown that $\int_0^x h(x)dx > \int_{tx}^{(t+1)x} h(x)dx$. (follows trivially for $x \geq 5$.) □

Now complete the counter-example:

$$\int_0^2 h(x)dx < 12+\frac{\delta}{2}+\{.01*10+(.015-\frac{3}{100\pi})\} < 14-2(.015-\frac{3}{100\pi}) = \int_3^5 h(x)dx$$

The first inequality follows from setting an upper bound on the sinusoidal perturbation introduced around 1.[17] Therefore, $f(5) > f(2) + f(3)$, which provides us with the counter-example to equation (7) and hence, to equation (5). In other words, as (6) does not imply (7), (4) does not imply (5), and hence, (4) does not imply (2).
That is, even if for all $t \in \mathbb{N}$ and for all $r \in (0,1)$: $g((1-r)^{t+1}) > g((1-r)^t)g((1-r))$ it does not imply that $\forall p, q \in (0,1)$: $g(pq) \geqslant g(p)g(q)$.

[16] $\delta = 50k\pi\cos(\pi/2 - .0001) = .157$ (approximately)
[17] This particular sinusoid dies down after $1.005 + \frac{\pi/2-.0001}{100\pi} < 1.01$ and never rises above the $h(x) = 1$ line by more than 6 units.

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