The Structure of Intertemporal Preferences under Uncertain Lifetime

Yoram Halevy*
Department of Economics
University of British Columbia
#997-1873 East Mall
Vancouver BC V6T 1Z1 CANADA
yhalevy@interchange.ubc.ca
http://www.econ.ubc.ca/halevy

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Abstract

Does experimental evidence of the *common difference effect* imply that intertemporal preferences are *time inconsistent*? Do these features of choice suggest inherent characteristics of preferences, or could they be traced to an underlying factor that shapes the environment?

The effect of uncertain lifetime on intertemporal preferences is studied. A demographic model that allows for unobservable heterogeneity in frailty (risk of mortality) accommodates the common difference effect, even in the presence of stationarity and time consistency. Furthermore, uncertain lifetime transforms any consumption path into a series of lotteries over consumption. If preferences are disproportionately sensitive to certainty, which is a key feature of present consumption, choices will be time inconsistent.

JEL Classification: D81, D91

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1 Introduction

The goal of this paper is to provide an alternative theoretical interpretation of existing experimental studies concerning choice over time. I study the effect of uncertainty, and especially how uncertain lifetime affects time preference. Uncertain lifetime should be interpreted throughout this paper as an allegory for situations in which planned consumption paths might not materialize. The importance of uncertain lifetime in determining temporal preferences has been acknowledged since Rae ([20], 1834), and in analyzing optimal consumption and saving problems since Yaari ([29], 1965). However, previous studies that accounted for uncertain lifetime did not have the vast experimental evidence focusing on intertemporal substitution, which is available today[1]. My focus in this paper is on the apparent tension between time consistency and experimental evidence.

Time consistency entails that when an individual can re-optimize, she does not have an incentive to deviate from her ex-ante decisions. Strotz [25] proved that in a deterministic model, time consistency is equivalent to exponential discounting of (certain) payoffs. In a preliminary result, I show that time consistency is unrelated to the stochastic process governing mortality: even if the environment is uncertain, time consistency is equivalent to exponential discounting.

Experimental studies have tested the descriptive applicability of the normative assumption of time consistency. The leading evidence brought against exponential discounting is the *common difference effect* (Loewenstein and Prelec [15]): people (as animals) are more sensitive to a given time delay in consumption if it occurs earlier rather than later. That is, as two dates are moved uniformly further into the future, the willingness to sacrifice later consumption for earlier consumption diminishes. This effect is present especially for short horizons (Fredrick, Loewenstein and O'Donoghue [7]). The experimental evidence has led many researchers to argue that the common difference effect provides an unambiguous proof that the intertemporal utility function exhibits decreasing discount rates[2]. Furthermore, anecdotal evidence of self control problems, testify that the normative time consistency

[1] An excellent summary of the historical background, experimental evidence and suggestions for interpretation is included in the book *Choice Over Time* edited by Loewenstein and Elster [14].

[2] This is why this experimental evidence has become to be known as "hyperbolic discounting."

principle is not applied, and should be replaced by alternative descriptions of behavior: naivete, commitment or sophistication (Pollak [17]).

I make the following observation: in experiments (and empirical studies), one can only observe marginal rates of intertemporal substitution. If uncertainty is present, then intertemporal substitution is composed of *time preference* (discounting) which stands for intertemporal substitution under full certainty, and *odds of realizing* consumption. I show that if preferences are stationary and time consistency is imposed (hence discounting of time distance is at a constant rate), the common difference effect translates into a decreasing hazard rate over the realization of consumption. Casual observation may indicate that many interactions (especially evolutionary) are characterized by a decreasing hazard rate.

This paper shows that when uncertainty concerns lifetime (and this is one uncertainty experiments cannot abstract from), reasonable demographic models are compatible with time-consistent choices and the common difference effect. The decision problem analyzed incorporates unobservable *frailty* (risk of mortality), and an increasing hazard rate of mortality conditional on frailty. The result is driven by a learning argument: an agent is born with a prior belief over her frailty. As time passes and the individual survives, her conditional expected frailty is decreasing. If learning is sufficient to cancel the increase in the hazard due to aging, the subjective mixture distribution exhibits decreasing hazard rate, and is consistent with the empirical evidence on the common difference effect.

Note that the above argument can be made even if the individual does not consciously acknowledge the uncertainty of lifetime as motivating her behavior. The economic tradition of analyzing decision-theoretic problems requires the environment to be described accurately and the choices to be observed by the modeler. The preferences derived should be consistent with both. I believe most psychologists would share this paradigm and would not require awareness of the underlying uncertainty. Furthermore, even for short horizons when the probability of death is negligible, how can an individual be absolutely certain that a promise for future prize will be fulfilled in certainty? Lifetime uncertainty captures this aspect of doubt, which leads to a conservative behavior.

Does the introduction of uncertainty resurrect confidence in time consistency as a descriptive assumption? Not necessarily. I acknowledge another well known experimental evidence in the context of choice under risk - the *certainty effect* (Kahneman and Tversky [12]): individuals overweight certain outcomes relative to very likely but not completely certain outcomes. Expected utility cannot accommodate these preferences, but most non-expected utility models will allow this pattern. The risk of mortality transforms any consumption path into a risky plan, since the consumption in each period will be realized only if the agent survives to that period. Current consumption has a unique feature: it is certain. I show, by means of example, that if an individual's behavior exhibits the *certainty effect*, her willingness to sacrifice consumption in the near future for present consumption would be very high, relative to her MRS between consumption at equally distant dates in the future. As time passes and the agent survives, she would revise her choices, since the former far away future becomes immediate and certain. Hence, relaxing expected utility, and in particular allowing for the certainty effect, results in observed choices that satisfy the common difference effect and exhibit time inconsistency together.

Reconciling exponential discounting with the common difference effect shows that this specific evidence does not contradict time-consistent preferences. However, the ample anecdotal evidence that points to the direction of self control problems and usage of commitment devices, is not refuted by the argument presented here. I present an avenue that exhibits these properties, where the cause of time inconsistency is the inherent uncertainty of any future consumption contrasted with the certainty of present consumption. Overweighting of certain (immediate) outcomes results in choice pattern that exhibits *preference reversal*.

If a model abstracts from uncertain lifetime, acknowledging the certainty effect serves as a foundation for a reduced form representations as in Loewenstein and Prelec [15] or Laibson [13], enabling the modeler to capture both time inconsistency and experimental evidence such as the common difference effect. Furthermore, it allows to draw a causality relation from choice under uncertainty to the observed behavior in choice over time.

The paper is organized as follows: Section 2 describes the environment facing the individual. Section 3 derives the necessary conditions for an optimal consumption path for every discount function. In Section 4 I show that Strotz's result generalizes to uncertain lifetime. That is, for stationary preferences, the agent's decisions are time consistent only if time distance is discounted exponentially. Section 5 uncovers the relation between the common difference effect and the decreasing hazard rate property. In Section 6 I show that if individual's frailty is unknown ex-ante, demographic models that include belief updating can accommodate decisions that exhibit the common difference effect. Section 7 re-introduces time inconsistency into the framework of uncertain lifetime, through relaxing the expected utility structure of preferences. All technical proofs appear in the Appendix.

2 The Environment

The problem analyzed here is an allocation problem of an individual with an unknown lifetime. For simplicity I abstract from all other uncertainties (e.g. income), which could be treated similarly (Yaari [29]). The ex-ante optimal program at time 0, and the optimal program conditional on living at time $t > 0$ are characterized. The agent's time of death is denoted by T. Since she does not know her time of death, T is a random variable with pdf $\pi(T)$ on $[0, \infty)$. The probability that the consumer will be alive at time s is given by the survival function $\Omega(s)$:

$$\Omega(s) = \int_s^{\infty} \pi(t)\, dt \tag{1}$$

The hazard rate at s, which is the pdf of T conditional on the agent living at time s, is given by:

$$r(s) = \frac{\pi(s)}{\Omega(s)} \tag{2}$$

Integrating both sides of (2), shows that the hazard rate fully characterizes the distribution of T by the following relation:

$$\Omega(s) = e^{-\int_0^s r(t)dt} \tag{3}$$

Let $\pi_t(s)$ where $t \leq s$ denote the conditional pdf of T at s given the consumer is alive at t :

$$\pi_t(s) = \frac{\pi(s)}{\Omega(t)} \tag{4}$$

and let $\Omega_t(s)$ be the probability the consumer will be alive at s conditional on being alive at t :

$$\Omega_t(s) = \int_s^\infty \pi_t(\tau)\, d\tau = \frac{\Omega(s)}{\Omega(t)} \tag{5}$$

The consumer has an endowment of $K(0)$ which she would like to allocate to consumption between 0 and T. Hence the problem is to find the optimal consumption at each period in time. Assume no depreciation, so the law of motion of the state variable K is given by:

$$\frac{dK(s)}{ds} = -C(s) \tag{6}$$

The consumer's intertemporal utility function is additive separable and stationary, when her instantaneous utility function is increasing and concave. She discounts future consumption by the discount function $\alpha(\cdot)$, which is a function of the time distance between the future and the present.

3 Optimization

Given this simple environment, the consumer's optimization problem at time t is:

$$\max \int_t^\infty \Omega_t(s)\, \alpha(s-t)\, u(c(s), s)\, ds \tag{7}$$

$$s.t.$$

$$\begin{cases} \frac{dK(s)}{ds} = -C(s) \\ K(t) > 0 \; given \\ K(s) \geq 0 \text{ and } C(s) \geq 0 \; \forall \; s \geq t \end{cases}$$

The standard optimality conditions are given by:

$$\Omega_t(s)\, \alpha(s-t)\, u'(c(s), s) = \lambda(s) \;\; \forall \; s \geq t \tag{8}$$

$$\frac{d\lambda(s)}{ds} = 0 \Longrightarrow \lambda(s) = \widehat{\lambda}_t$$

The ex-ante planning problem is characterized by substituting $t = 0$ into (7), and the optimality conditions:

$$\Omega(s)\,\alpha(s)\,u'(c(s), s) = \widehat{\lambda}_0 \;\forall\; s \geq 0 \tag{9}$$

4 Time Consistency

I follow Strotz [25] in characterizing the discount function for which the consumer's choices at time t will abide by her original plan. That is, for which $\alpha(\cdot)$ the consumer does not have an incentive to deviate ex-post from her original plan? The difference from Strotz is that here the consumer does not know her time of death, hence she might die before exhausting all of $K(0)$, a situation which is not ex-post optimal.

Theorem 1 *The consumer's decisions are time consistent if and only if she discounts time exponentially, that is:*

$$\alpha(t) = Ae^{-\delta t} \quad for\ A > 0\ and\ \delta \in \Re$$

Proof. See Appendix. ■

Hence, Strotz's [25] result survives uncertain lifetime. If the consumer is impatient, then $\alpha(\cdot)$ is a non-increasing function, and hence $\delta \geq 0$. Note that time consistency says nothing about the mortality process. In particular, it does not imply the constant hazard rate property as previously suggested in the literature[3].

5 The Common Difference Effect

When uncertainty concerning the realization of consumption exists, as in the case of uncertain lifetime, the observed marginal rate at which an individual is willing to substitute consumption between two periods is composed of pure time preferences and belief concerning survival. Assuming the instantaneous

[3] E.g. Blanchard [4] fn. 4

utility function is stationary (that is: $u(c(s), s) = u(c(s))$ for every s), the marginal rate of substitution of consumption at t for consumption at $t+\tau$, is given by (using (3)):

$$\begin{aligned} MRS_{t,t+\tau} &= \frac{u'(c(t))\, e^{-\delta t}\Omega(t)}{u'(c(t+\tau))\, e^{-\delta(t+\tau)}\Omega(t+\tau)} = \qquad (10) \\ &= \frac{u'(c(t))\, e^{-\delta t} e^{-\int_0^t r(s)ds}}{u'(c(t+\tau))\, e^{-\delta(t+\tau)} e^{-\int_0^{t+\tau} r(s)ds}} = \frac{u'(c(t))}{u'(c(t+\tau))} e^{\delta\tau + \int_t^{t+\tau} r(s)ds} \end{aligned}$$

If we calculate the marginal rate of intertemporal substitution along a constant baseline consumption level $\bar{c}$ in all periods[4] then:

$$MRS_{t,t+\tau}\big|_{c(t)=c(t+\tau)} = e^{\delta\tau + \int_t^{t+\tau} r(s)ds} \qquad (11)$$

Experiments have shown (for a recent survey of results see Fredrick, Loewenstein and O'Donoghue [7]) that this function is decreasing in t. That is, the rate at which an individual is willing to substitute consumption in t for consumption in $t+\tau$ is a decreasing function of t. This is what Loewenstein and Prelec [15] call the "common difference effect" and commonly described as "hyperbolic discounting." In the presence of uncertainty, the MRS is composed of time discounting ($e^{\delta\tau}$) and the inverse of the probability the agent will be alive at $t+\tau$ conditional on being alive at t $\left(\frac{1}{\Omega_t(t+\tau)} = \frac{\Omega(t)}{\Omega(t+\tau)}\right)$. Since time is discounted at a constant rate, the evolution of $\Omega(\cdot)$ determines the path of the MRS.

Proposition 1 *If the consumer is time consistent then the Marginal Rate of Intertemporal Substitution of consumption at t for consumption at $t+\tau$ along a baseline consumption level is a decreasing function of t (the common difference effect) if and only if the uncertainty about lifetime has the decreasing hazard rate property.*

Proof. See Appendix. ■

[4] This follows the tradition of Fisher [8]. Identical results could be supported if one follows Uzawa [26] and defines time preference as the MRS of time t *utility* for time $t+\tau$ *utility.* For a discussion of alternative definitions of time preference see Becker and Mulligan [3].

Assume the consumer is comparing consumption of ten apples today to the consumption of a dozen apples in a year. Although in the comparison itself there is no explicit uncertainty, all real life decisions involve uncertainty. In the simplest case (presented above), the consumer does not know whether she will be alive in a year. There is some probability she might die beforehand, and will not be able to enjoy the promised apples. This captures the notion that even if the decision maker lives a year, there is some risk the apples will not be available. This reasoning might be motivated by the common wisdom that the further away in time is the promise, the lower is the probability it will be fulfilled. Thus, even if the probability of actual death is negligible, the agent might think she is facing risk on the payment side. The willingness to sacrifice later consumption for an earlier consumption might change when the time horizon changes. In comparing ten apples in ten years to a dozen in eleven years, the decision maker might make the following argument: "Conditional on surviving ten years, the probability of surviving an extra year is higher than the probability of surviving a year today." On the dual (payment) side: "Conditional on the promise of ten apples in ten years being kept, the probability that the promise of a dozen apples in eleven years will be honored, is higher than the prior subjective probability that the dozen apples will be delivered in a year." This result may be motivated from the matching technology we are faced with in everyday life. For example, if I just met a new acquaintance, the probability I will know her whereabouts in a week is lower than the probability I will know her whereabouts in a year and a week, conditional on knowing her whereabouts in a year. One might argue that individuals have adapted to this matching environment, and in their answers to experiments cannot abandon this rule of thumb. This reasoning relies on some inertia which is present in the decision maker's decision process. She cannot adjust her hard-wired decision rule to the environment presented at the experiment. However, this paper does not rely on such bounded-rationality argument, since only in very special circumstances can uncertain lifetime be assumed away in an intertemporal decision problem.

It might be helpful to have a different view of the previous result. According to (10), the rate of change of the marginal rate of substitution of current consumption for future consumption at t, when $c(0) = c(t)$, is given by:

$$\frac{\frac{dMRS_{0,t}|_{c(0)=c(t)}}{dt}}{MRS_{0,t}|_{c(0)=c(t)}} = \delta + r\left(t\right) \tag{12}$$

Hence, the rate at which current consumption could be substituted for future consumption is increasing with a diminishing rate if and only if the hazard rate, $r(t)$, is decreasing.

6 Demographic Model of Uncertain Lifetime

As shown in the previous section, individuals are more sensitive to a given time delay if it occurs earlier rather than later, if and only if the hazard rate of mortality is decreasing. However, it has been long suggested (at least since Gompertz in the first half of the 19th century) that for an individual, the hazard rate of death is increasing in age. These demographic models are based on an assumption of a homogeneous population. Following the demographic literature (Vaupel et al. [27] and see Namboodiri and Suchindran [16] for a survey), I argue here that if individuals differ in their *frailty* (force of mortality), and when born have a prior belief over the frailty component of their hazard rate, allowing Bayesian updating of this prior may lead to a behavior consistent with a decreasing hazard rate, although the actual hazard rate (conditional on the true frailty) may be increasing.

Let (T, Θ) be a bivariate random variable. Frailty is represented by the non-negative random variable - Θ, which represents the consumer's endowment of longevity. The individual holds a prior belief over the distribution of Θ, denoted by the absolutely continuous cdf F and the pdf f. Conditional on $\Theta = \theta$, the pdf of T is given by $\pi\left(s|\theta\right)$ and the probability that the individual will be alive at time s is given by $\Omega\left(s|\theta\right) = \int_s^\infty \pi\left(t|\theta\right) dt$. Thus lifetime has a mixture distribution, and the unconditional survival function is given by:

$$\bar{\Omega}\left(s\right) = \int \Omega\left(s|\theta\right) dF\left(\theta\right) \tag{13}$$

The following Proposition, shows that Strotz's result survives this extended model of unknown frailty:

Proposition 2 *Let (T,Θ) be a bivariate random variable denoting time of death and endowment of frailty, respectively. Denote by $F(\cdot)$ the absolutely continuous cdf of the prior belief over Θ, and by $\pi(\cdot|\theta)$ the conditional pdf of T given $\Theta=\theta$. The consumer's decisions are time consistent if and only if she discounts time exponentially, that is:*

$$\alpha(t) = Ae^{-\delta t} \text{ for } \delta \in \Re \text{ and } A > 0$$

Proof. See Appendix ■

As before, $\delta > 0$ represents impatience.
I follow the demographic literature (that follows Cox [5]) and assume the hazard rate is multiplicatively dependent on the frailty θ:

$$r(t,\theta) = \theta\rho(t) \tag{14}$$

where $\rho(t)$ is the component of the hazard rate which is time dependent. In what follows, I will construct an example that assumes a reasonable prior belief over frailty and an increasing hazard rate conditional on frailty. Those define a well-behaved mixture distribution. Conditions for a decreasing hazard rate for the mixture distribution will be derived.

6.1 A Gamma Prior Belief over Frailty

I follow Vaupel, Manton and Stallard [27] and assume frailty at birth is gamma distributed[5] with pdf:

$$f(\theta) = \frac{\gamma^k}{\Gamma(k)}\theta^{k-1}e^{-\gamma\theta} \text{ for } \theta > 0 \tag{15}$$

where γ and k are parameters of the distribution, such that $E(\theta) = \bar{\theta} = \frac{k}{\gamma}$ and $Var(\theta) = \frac{k}{\gamma^2}$. This distribution is chosen because of its analytical tractability and flexibility. The following proposition characterizes the evolution of the conditional frailty, independently of the distribution of the conditional hazard rate.

[5] Weitzman [28] uses the gamma distribution to capture the distribution of the discount factor in the population.

Proposition 3 *If the prior belief over frailty has a gamma distribution then the posterior frailty conditional on surviving t is:*

$$\theta \,|\{T \geq t\} \sim Gamma\,(k, \gamma\,(t))$$

where $\gamma\,(t) = \gamma + \int_0^t r\,(s)\,ds$

Proof. See Appendix. ■

The consumer is born and does not know her true frailty. She only has a prior belief over it. As she ages, she learns about her true frailty in the following way: if she had been frail (high θ), the probability of a short lifetime would be high. Therefore, as t increases, the probability that she has a low frailty increases, as summarized by the expected value of the conditional distribution of frailty which is equal to $\frac{k}{\gamma(t)}$.

6.2 Gompertz's (increasing) Conditional Hazard

Now assume that the time-dependent component of the hazard rate follows Gompertz's rule:

$$\rho\,(t) = e^{bt} \text{ where } b \geq 0 \tag{16}$$

That is, conditional on frailty, the hazard rate is increasing exponentially. Even now, if $b\gamma < 1$, the mixture distribution has a decreasing hazard rate and the marginal rate of intertemporal substitution is diminishing in time.

Proposition 4 *Assume that conditional on the frailty value, the hazard rate of death follows Gompertz's law and the prior belief over frailty is gamma(k, γ). If $\gamma b < 1$, then a consumer with time-consistent preferences exhibits the common difference effect.*

Proof. See Appendix. ■

To gain intuition of this result, normalize k to 1 (exponential prior). Then the condition is $b < \frac{1}{\gamma} = \bar{\theta}\,(0)$. Thus, the individual believes ex-ante that her frailty is higher than the rate of change of the hazard rate. Then, as time progresses and she survives, she will update her subjective frailty sufficiently to cause the subjective hazard rate (of the mixture distribution) to decrease.

Although the focus of this paper is theoretical, it is worthwhile to note some partial evidence in support of this explanation. Hurd, McFadden and Merrill [11] study the Asset and Health Dynamics Among the Oldest-Old (AHEAD). They report (tables 14 and 15) that for the population of 70-89, the subjective survival probability decreases much more moderately than the life table survival rates. I conjecture that this finding is due to an extensive subjective updating of frailty during those years.

6.2.1 A Constant Hazard and Related Literature

Although a model that assumes a constant hazard rate cannot be supported empirically, it sheds light on the evolution of conditional frailty, and could be applied easily to add heterogeneity within a generation in an overlapping generation model like Blanchard's [4]. Assume frailty is the only component of the hazard rate. That is, the hazard would be independent of age (constant hazard rate) and normalized to 1. Then, according to Proposition 3 the conditional distribution of frailty would be $gamma\left(k, \gamma + t\right)$. It is easy to show that in this case:

$$\bar{\Omega}_t\left(s\right) = \left(\frac{\gamma + t}{\gamma + s}\right)^k$$

Hence, the marginal rate of intertemporal substitution between consumption at t and $t + \tau$ when $c\left(t + \tau\right) = c\left(t\right)$ is:

$$MRS_{t,t+\tau}\big|_{c(t)=c(t+\tau)} = e^{\delta\tau}\left(1 + \frac{\tau}{\gamma + t}\right)^k \tag{17}$$

which clearly is a decreasing function of t (since $\gamma b = 0$) and has the explicit hyperbolic structure suggested in the Psychological literature.

Sozou [24] presents an example where there is no time discounting, utility is the identity function and the hazard rate is constant but unknown to an animal[6]. He shows that the conditional hazard is decreasing in time.

[6] Hence usually the optimal consumption stream would be a corner solution.

Azfar [2] discusses a case of unknown constant hazard, and shows that the apparent discount rate is diminishing, but decisions are time consistent. The results in this paper show that the conditional hazard could be increasing, as long as the mixture distribution of lifetime has the decreasing hazard rate property. Furthermore, Proposition 2 shows that as long as time discounting is exponential, time consistency will prevail and is independent of the hazard rate. The next section uncovers a path of re-introducing time inconsistency into the analysis of uncertain payoffs.

7 Why Time Inconsistency?

Up to now, this paper has suggested evidence that the common difference effect might be related to the uncertainty of future consumption. I have shown that diminishing marginal rate of intertemporal substitution may be supported by demographic models that allow unobservable heterogeneous frailty, about which the agent learns as time passes. As noted earlier, the argument is not constrained to uncertain lifetime, and is applicable to any decision problem where the payoff is subject to random process with a (subjective) decreasing hazard rate property.

What about time inconsistency? the previous discussion showed that the common difference effect evidence is inconclusive, and does not necessarily imply time inconsistency even in the presence of stationarity. This section shows, by means of example, that uncertainty might generate the observed time inconsistency in intertemporal choices. For simplicity of the argument, assume no time discounting, a homogeneous population with unitary frailty and a constant hazard of mortality. Consider the following thought experiment comprised of two choices. The subject is asked to choose between:

- \$100 now or \$110 in a month;
- \$100 in two years or \$110 in two years and one month.

Most of the subjects would prefer \$100 now in the first problem and \$110 in the second. This is consistent with the common difference effect, but does not imply it[7]. Assume that the subjective hazard rate per month is 1%[8].

[7] The common different effect claims the MRS is monotonically decreasing.

[8] This is done for expositional purposes only. If there is time discounting (in addition to uncertainty over the realization of the payment), the hazard rate could be lowered drastically.

The two alternatives translate into a choice between two prospects:

- $(\$100, 1)$ or $(\$110, 99\%)$
- $(\$100, 78.57\%)$ or $(\$110, 77.78\%)$ [9]

where prospect $(\$x, p)$ represent a lottery with probability p of winning $\$x$ and probability $1 - p$ of winning \$0. It has been well documented in experiments (Kahneman and Tversky [12]) that although expected utility predicts having the same choice (\$100 or \$110) in both, many subjects exhibit the *certainty effect:* they overweight certain outcomes relative to very likely but not completely certain outcomes. As a result, they prefer \$100 in the first choice and \$110 in the second choice. This translates to \$100 today (at $t = 0$) and \$110 in two years and a month. Contrary to the discussion in prior sections, when the certainty effect is present, the choices would *not* be time consistent: although today (at $t = 0$) a subject prefers \$110 in two years and a month, her preferences in two years (conditional on surviving) are to prefer \$100 immediately (at $t = 2$ years) over \$110 in a month (at $t = 2$ years and a month)[10]. These choices are often referred to as *preference reversal*[11]. The source for the certainty effect might be Rubinstein's ([21], [22]) similarity between the probabilities of realizing the prize in twenty four and twenty five months, while receiving the prize with certainty today is not perceived as similar to the probability of receiving it in a month. Although the certainty effect does not create necessarily a monotone common

[9] These prospects are derived from:

$$\left(\$100, 0.99^{24}\right) \text{ or } \left(\$110, 0.99^{25}\right)$$

[10] Since the hazard rate is constant at 1% a month, there is no place for learning, and the choice at $t = 2$ years between:

$$(\$100, \textit{today}) \ \textit{or} \ (\$110, \textit{in a month})$$

translates into a choice between:

$$(\$100, 1) \ \textit{or} \ (\$110, 99\%)$$

and if the certainty effect is present, the subject prefers \$100 at $t = 2$, contrary to her preferences at $t = 0$.

[11] The analysis assumes preferences are consequential. That is, the decision in two years ignores the uncertainty borne by the subject up to that point in time.

difference effect, it does create a pointwise similar effect when comparing the present to the future. That is, the willingness to sacrifice later consumption for earlier consumption is highest in the present[12]. Hence it fits the experimental evidence that the common difference effect is present especially for short horizons. In this sense, the approach here may serve as a foundation for the quasi-hyperbolic (δ, β) parameterization (Laibson, [13]), which assumes that the MRS between consumption at two consecutive periods is highest in the present for the immediate future $(\beta\delta)$, and from then on it is constant at β.

8 Concluding Remarks

This paper shows that introducing uncertainty not only allowed us to view existing results in a different light, but related two choice problems: choice under risk and choice over time, and enabled the former to illuminate the latter. It was shown that both attributes of the empirical preferences: the common difference effect and time inconsistency, could be traced theoretically to uncertain lifetime, and neither of them is an unambiguous evidence of the other. Although this paper tried to give new answers to existing problems, it created many new questions to be answered in future research. For example, could surveys of expected longevity (as studied in Smith, Taylor and Sloan [23], Hamermesh [10] and Hurd et al [11]) confirm the demographic model? How well do certain non-expected utility functions that are consistent with the certainty effect (e.g. Epstein and Zin [6]), accommodate intertemporal choice patterns? How similar are the consumption paths induced by naivete, commitment or sophistication (Pollak [17], Laibson [13]) with a known horizon to those induced by non-expected utility functions with stochastic horizon that are dynamically inconsistent, resolute choice (non-consequentialist) or recursive [6] respectively?

[12] The similarity between the two paradigms of choice under risk and over time was noted before by Rachlin et al [19], Gilboa [9], Loewenstein and Prelec [15], Quiggin and Horowitz [18] and Rubinstein [22]. However, to my best knowledge, this is the first study to draw a causality relation between the two due to uncertain lifetime.

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A Proofs

Theorem 1 *The consumer's decisions are time consistent if and only if she discounts time exponentially, that is:*

$$\alpha(t) = Ae^{-\delta t} \quad for\ A > 0\ and\ \delta \in \Re$$

Proof. The consumption at time s as planned at t should be equal to the original plan for time s consumption:

$$u'(c(s), s) = \frac{\widehat{\lambda}_0}{\Omega(s)\,\alpha(s)} = \frac{\widehat{\lambda}_t}{\Omega_t(s)\,\alpha(s-t)}$$

$$\alpha(s) = \frac{\widehat{\lambda}_0}{\widehat{\lambda}_t}\frac{\Omega_t(s)}{\Omega(s)}\alpha(s-t) \tag{18}$$

$$\frac{\Omega_t(s)}{\Omega(s)} = \frac{\Omega(s)/\Omega(t)}{\Omega(s)} = \frac{1}{\Omega(t)}$$

Letting $a(t) := \frac{\widehat{\lambda}_0}{\widehat{\lambda}_t}\frac{1}{\Omega(t)}$ and $\tau := s - t$, this is a functional equation in $\alpha(\cdot)$:

$$\alpha(t+\tau) = a(t)\,\alpha(\tau)$$

Taking logarithm and setting $f(\cdot) := \ln(\alpha(\cdot))$ and $h(\cdot) := \ln(a(t))$, one gets:

$$f(t+\tau) - f(\tau) = h(t)$$

Setting $\tau = 0$: $h(t) = f(t) - f(0)$ hence:
$f(t+\tau) - f(\tau) = f(t) - f(0)$. Letting $\kappa := -f(0)$:
$f(t+\tau) - f(\tau) - f(t) = \kappa$ or by adding and subtracting κ from the lhs:

$$[f(t+\tau)+\kappa] - [f(\tau)+\kappa] - [f(t)+\kappa] = 0$$

Let $\varphi(\cdot) = f(\cdot) + \kappa$. The following functional equation:

$$\varphi(\tau+\tau) = \varphi(\tau) + \varphi(t) \quad \text{for all } t, \tau \geq 0 \tag{19}$$

is Cauchy's basic equation, which, if $\varphi(\cdot)$ is continuous at a point, is uniquely solved by:

$$\varphi(t) = ct \quad \forall t \geq 0$$

(Aczél [1]). It is immediate from (19) that $\varphi(0) = 0$. Hence $\alpha(t) = e^{f(t)} = e^{\varphi(t)-\kappa} = e^{ct+\ln(\alpha(0))} = \alpha(0)\,e^{ct}$, letting $A = \alpha(0)$:

$$\alpha(t) = Ae^{ct}$$

Letting $\delta = -c$ the theorem is proved. ■

Proposition 1 *If the consumer is time consistent then the Marginal Rate of Intertemporal Substitution of consumption at t for consumption at $t+\tau$ along a baseline consumption level is a decreasing function of t (the common difference effect) if and only if the uncertainty about lifetime has the decreasing hazard rate property.*

Proof.

$$\left.\frac{dMRS_{t,t+\tau}}{dt}\right|_{c(t)=c(t+\tau)} = e^{\delta\tau+\int_t^{t+\tau} r(s)ds}\left[r\left(t+\tau\right)-r\left(t\right)\right]$$

Hence:

$$sign\left(\left.\frac{dMRS_{t,t+\tau}}{dt}\right|_{c(t)=c(t+\tau)}\right) = sign\left(r\left(t+\tau\right)-r\left(t\right)\right)$$

■

Proposition 2 *Let (T,Θ) be a bivariate random variable denoting time of death and endowment of frailty, respectively. Denote by $F(\cdot)$ the absolutely continuous cdf of the prior belief over Θ, and by $\pi(\cdot|\theta)$ the conditional pdf of T given $\Theta=\theta$. The consumer's decisions are time consistent if and only if she discounts time exponentially, that is:*

$$\alpha\left(t\right)=Ae^{-\delta t}\textit{for } \delta\in\Re \textit{ and } A>0$$

Proof. This problem has a similar structure to the one studied in Theorem 1. The updated pdf of Θ conditional on surviving to t is given by:

$$f_t\left(\theta\right)=f\left(\theta\left|T\geq t\right.\right)=\frac{f\left(\theta\right)\Omega\left(t|\theta\right)}{\Omega\left(t\right)}$$

Hence, the updated survival function:

$$\begin{aligned}\Omega_t\left(s\right) &= \Pr\left\{T\geq s\left|T\geq t\right.\right\}=\int\Omega_t\left(s|\theta\right)dF_t\left(\theta\right)= \\ &= \int\frac{f\left(\theta\right)\Omega\left(t|\theta\right)}{\Omega\left(t\right)}\Omega_t\left(s|\theta\right)=\frac{1}{\Omega\left(t\right)}\int\Omega\left(s|\theta\right)dF\left(\theta\right)=\frac{\Omega\left(s\right)}{\Omega\left(t\right)}\end{aligned}$$

Therefore, the functional equation (18) may be reduced similarly to Cauchy's basic functional equation. ■

Proposition 3 *If the prior belief over frailty has a gamma distribution, then the posterior frailty conditional on surviving t is:*

$$\theta\left|\{T\geq t\}\right.\sim Gamma\left(k,\gamma\left(t\right)\right)$$

where $\gamma\left(t\right)=\gamma+\int_0^t\rho\left(s\right)ds$

Proof. Note that under the multiplicative hazard:

$$\Omega\left(s|\theta\right)=e^{-\int_0^s r(t,\theta)dt}=e^{-\int_0^s\theta\rho(t)dt}=e^{-\theta\int_0^s\rho(t)dt}=e^{-\theta H(s)}=\left[\Omega\left(s\right)\right]^{\theta}$$

where:

$$H\left(s\right)=\int_0^s\rho\left(t\right)dt$$

Now:

$$f\left(\theta|T\geq t\right)=\frac{f\left(\theta,T\geq t\right)}{\bar{\Omega}\left(t\right)}=\frac{\Omega\left(t|\theta\right)f\left(\theta\right)}{\bar{\Omega}\left(t\right)}=\frac{e^{-\theta H(t)}\frac{\gamma^k}{\Gamma(k)}\theta^{k-1}e^{-\gamma\theta}}{\bar{\Omega}\left(t\right)}$$

Find $\bar{\Omega}\left(t\right)$ by the normalization:

$$\frac{1}{\bar{\Omega}\left(t\right)}\int_0^{\infty}e^{-\theta H(t)}\frac{\gamma^k}{\Gamma\left(k\right)}\theta^{k-1}e^{-\gamma\theta}d\theta=1$$

Hence:

$$\begin{aligned}\bar{\Omega}\left(t\right) &= \frac{\gamma^k}{\Gamma\left(k\right)}\int_0^{\infty}e^{-\theta(\gamma+H(t))}\theta^{k-1}d\theta=\frac{\gamma^k}{\Gamma\left(k\right)}\frac{1}{\gamma\left(t\right)^{k-1}}\int_0^{\infty}e^{-\theta\gamma(t)}\left[\theta\gamma\left(t\right)\right]^{k-1}d\theta= \\ &= \frac{\gamma^k}{\Gamma\left(k\right)}\frac{1}{\gamma\left(t\right)^k}\int_0^{\infty}e^{-\xi}\xi^{k-1}d\xi=\frac{\gamma^k}{\Gamma\left(k\right)}\frac{1}{\gamma\left(t\right)^k}\Gamma\left(k\right)=\left(\frac{\gamma}{\gamma\left(t\right)}\right)^k\end{aligned}$$

Substituting back:

$$f(\theta|T \geq t) = \frac{e^{-\theta(\gamma+H(t))}\frac{\gamma^k}{\Gamma(k)}\theta^{k-1}}{\left(\frac{\gamma}{\gamma(t)}\right)^k} = \frac{\gamma(t)^k}{\Gamma(k)}e^{-\theta\gamma(t)}\theta^{k-1}$$

The distribution of frailty conditional on surviving to age t is gamma$(k, \gamma(t))$
■

Proposition 4 *Assume that conditional on the frailty value, the hazard rate of death follows Gompertz's law and the prior belief over frailty is gamma(k, γ). If $\gamma b < 1$, then a consumer with time-consistent preferences exhibits the common difference effect.*

Proof. Define the cumulative hazard function $H(t)$ by:

$$H(t) = \int_0^t \rho(t)\,dt = \int_0^t e^{bs}ds = \frac{1}{b}\left[e^{bt} - 1\right]$$

Let $\bar{r}(t)$ denote the expected value of the hazard rate conditional on living at least t, and let $\bar{\theta}(t)$ be the expected value of frailty conditional on living at least t. Then:

$$\bar{r}(t) = \rho(t)\,\bar{\theta}(t) = e^{bt}\frac{k}{\gamma + H(t)} = \frac{ke^{bt}}{\gamma + \frac{1}{b}\left[e^{bt} - 1\right]}$$

It is easy to see that $\frac{d\bar{r}(t)}{dt} < 0$ if and only if $b\gamma < 1$. Hence, the mixture distribution possesses a diminishing hazard rate and Proposition 1 applies.
■