Chapter 8

EXACT AND SUPERLATIVE INDEX NUMBERS*

W.E. Diewert

1. Introduction

One of the most troublesome problems facing national income accountants and econometricians who are forced to construct some data series, is the question of which functional form for an index number should be used. In the present paper, we consider this question and relate functional forms for the underlying production or utility function (or aggregator function, to use a neutral terminology) to functional forms for index numbers.

First, define a quantity index between periods 0 and 1, \( Q(p^0, p^1, x^0, x^1) \), as a function of the prices in periods 0 and 1, \( p^0 > 0_N \) and \( p^1 > 0_N \) (where \( 0_N \) is an \( N \) dimensional vector of zeros), and the corresponding quantity vectors, \( x^0 > 0_N \) and \( x^1 > 0_N \). A price index between periods 0 and 1, \( P(p^0, p^1, x^0, x^1) \), is a function of the same price and quantity vectors. Given either a price index or a quantity index, the other function can be defined implicitly by the following equation (Fisher’s [1922] weak factor reversal test):

\[
(1.1) \quad P(p^0, p^1, x^0, x^1) Q(p^0, p^1, x^0, x^1) = p^1 \cdot x^1 / p^0 \cdot x^0;
\]

i.e., the product of the price index times the quantity index should yield the expenditure ratio between the two periods. (We indicate the inner product of two vectors as \( p \cdot x \) or \( p^T x \).)

Examples of price indexes are

\[
\begin{align*}
P_{La}(p^0, p^1, x^0, x^1) &\equiv p^1 \cdot x^0 / p^0 \cdot x^0 \quad \text{(Laspeyres price index),} \\
P_{Pa}(p^0, p^1, x^0, x^1) &\equiv p^1 \cdot x^1 / p^0 \cdot x^1 \quad \text{(Paasche price index).}
\end{align*}
\]

*This article was first published in the Journal of Econometrics 4(2), 1976, pp. 115–145. A preliminary version was presented at Stanford in August 1973. The author is indebted to L.J. Lau, D. Aigner, K.J. Arrow, E.R. Berndt, C. Blackorby, L.R. Christensen and K. Lovell for helpful comments. This research was partially supported by National Science Foundation Grant GS-3269-A2 at the Institute for Mathematical Studies in the Social Sciences at Stanford University, and by the Canada Council.
The geometric mean of the Paasche and Laspeyres indexes has been suggested as a price index by Bowley [1928] and Pigou [1912], but it is Irving Fisher [1922] who termed the resulting index ideal:

\[ P_{id}(p^0, p^1, x^0, x^1) \equiv (p^1 \cdot x^0 p^{-1} \cdot x^1/p^0 \cdot x^0 p^{-1} \cdot x^1)^{1/2}. \]

The Laspeyres, Paasche and ideal quantity indexes are defined in a similar manner — quantities and prices are interchanged in the above formulae. In particular, the ideal quantity index is defined as

\[ Q_{id}(p^0, p^1, x^0, x^1) \equiv (p^1 \cdot x^1 p^{-1} \cdot x^0/p^0 \cdot x^0 p^{-1} \cdot x^0)^{1/2}. \]

Notice that \( P_{id}Q_{id} = p^1 \cdot x^1/p^0 \cdot x^0 \); i.e., the ideal price and quantity indexes satisfy the ‘adding up’ property (1.1). The following theorem shows that the ideal quantity index may be used to compute the quantity aggregates \( f(x^r) \) provided that the aggregator function \( f \) has a certain functional form.

**Theorem 1.4.** (Byushgens [1925], Konius and Byushgens [1926], Frisch [1936; 30], Wald [1939; 331], Afriat [1972b; 45] and Pollak [1971a]): Let \( p^r \gg 0 \) for periods \( r = 0, 1, 2, \ldots, R \), and suppose that \( x^r > 0N \) is a solution to \( \text{max}_{x} \{f(x) : p^r \cdot x \leq p^r \cdot x^r, x \geq 0N\} \), where \( f(x) \equiv (x^T A x)^{1/2} \equiv (\sum_{i=1}^{N} \sum_{k=1}^{N} x_k a_{ik} x_k)^{1/2}, a_{jk} = a_{kj} \), and the maximization takes place over a region where \( f(x) \) is concave and positive (which means \( A \) must have \( N - 1 \) zero or negative eigenvalues and one positive eigenvalue). Then

\[ f(x^r)/f(x^0) = Q_{id}(p^0, p^r, x^0, x^r), \quad r = 1, 2, \ldots, R. \]

Thus given the base period normalization \( f(x^0) = 1 \), the ideal quantity index may be used to calculate the aggregate \( f(x^r) = (x^r^T A x^r)^{1/2} \) for \( r = 1, 2, \ldots, R \), and we do not have to estimate the unknown coefficients in the \( A \) matrix. This is the major advantage of this method for determining the aggregates \( f(x^r) \) (as opposed to the econometric methods suggested by Arrow [1974]), and it is particularly important when \( N \) (the number of goods to be aggregated) is large compared to \( R \) (the number of observations in addition to the base period observation \( p^0 \), \( x^0 \)).

If a quantity index \( Q(p^0, p^r, x^0, x^r) \) and a functional form for the aggregator function \( f \) satisfy (1.5) then we say that \( Q \) is exact for \( f \). Konius and Byushgens [1926] show that the geometric quantity index \( \prod_{i=1}^{N} (x^1_i/x^0_i)^{s_i} \) (where \( s_i \equiv p^0_i x^0_i/p^0 \cdot x^0 \) is exact for a Cobb–Douglas aggregator function, while Afriat [1972b], Pollak [1971a] and Samuelson–Swamy [1974] present other examples of exact index numbers. However, it appears that out of all the exact index numbers thus far exhibited, only the ideal index corresponds to a functional form for \( f \) which is capable of providing a second order approximation to an arbitrary twice differentiable linearly homogeneous function. For a proof that the functional form \( (x^T A x)^{1/2} \) can provide such a second order approximation, see Diewert [1974b].

Let us call a quantity index \( Q \) ‘superlative’ (see Fisher [1922; 247] for an undefined notion of a superlative index number) if it is exact for \( f \) which can provide a second order approximation to a linearly homogeneous function.

In the following section, we show that the Törnqvist [1936], Theil [1965] and Kloeck [1966] [1967] quantity index (which has been used by Christensen and Jorgenson [1970], Star [1974], Jorgenson and Griliches [1972; 83], Star and Hall [1976] as a discrete approximation to the Divisia [1926] index) is also a superlative index number. In Section 3, we use the results of Section 2 to provide a rigorous interpretation of the Jorgenson–Griliches method of measuring technical progress for discrete data.

In Section 4, we introduce an entire family of superlative index numbers. Section 5 presents some conclusions which tend to support the use of Fisher’s ideal quantity index in empirical applications and the final section is an appendix which sketches the proofs of various theorems developed in the following sections.

2. The Törnqvist–Theil ‘Divisia’ Index and the Translog Function

Before stating our main results, it will be necessary to state a preliminary result which is extremely useful in its own right. Let \( z \) be an \( N \) dimensional vector and define the quadratic function \( f(z) \) as

\[ f(z) = a_0 + a^T z + \frac{1}{2} z^T A z \]

\[ = a_0 + \sum_{j=1}^{N} a_z z_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} z_i z_j, \]

where the \( a_i, a_{ij} \) are constants and \( a_{ij} = a_{ji} \) for all \( i, j \).

The following lemma is a global version of the Theil [1967; 222–223] and Kloeck [1966] local result.

**Lemma 2.2.** Quadratic Approximation Lemma. If and only if the quadratic function \( f \) is defined by (2.1), then

\[ f(z^1) - f(z^0) = \frac{1}{2} \| \nabla f(z^1) + \nabla f(z^0) \|^2 (z^1 - z^0), \]

where \( \nabla f(z^r) \) is the gradient vector of \( f \) evaluated at \( z^r \).
The above result should be contrasted with the usual Taylor series expansion for a quadratic function,
\[
f(z^1) - f(z^0) = [\nabla f(z^0)]^T(z^1 - z^0) + \frac{1}{2}(z^1 - z^0)^T \nabla^2 f(z^0)(z^1 - z^0),
\]
where $\nabla^2 f(z^0)$ is the matrix of second order partial derivatives of $f$ evaluated at an initial point $z^0$. In the expansion (2.3), a knowledge of $\nabla^2 f(z^0)$ is not required, but a knowledge of $\nabla f(z^1)$ is required. It must be emphasized that (2.3) holds as an equality for all $z^1$, $z^0$ belonging to an open set if and only if $f$ is a quadratic function.

Actually, the Quadratic Approximation Lemma (2.2) is closely related to the following result which we will prove as a corollary to (2.2):

**Lemma 2.4.** (Bowley [1928; 224–225]): If a consumer’s preferences can be represented by a quadratic function $f$ defined by (2.1), if $x^1 \gg 0_N$ is a solution to the utility maximization problem
\[
\text{max}_z \{f(z) : p^1 \cdot z = Y^1, \ z \geq 0_N\},
\]
where $p^1 \gg 0_N$ and $Y^1 \equiv p^1 \cdot x^1$, and if $x^0 \gg 0_N$ (i.e., each component of $x^0$ is positive) is a solution to the utility maximization problem
\[
\text{max}_z \{f(z) : p^0 \cdot z = Y, \ z \geq 0_N\},
\]
where $p^0 \gg 0_N$ and $Y^0 \equiv p^0 \cdot x^0$, then the change in utility between periods 0 and 1 is
\[
f(x^1) - f(x^0) = \frac{1}{2}(\lambda^*_i p^1 + \lambda^*_i p^0) \cdot (x^1 - x^0),
\]
where $\lambda^*_i$ is the marginal utility of income in period $i$ for $i = 0, 1$. That is, $\lambda^*_i$ is the optimal value of the Lagrange multiplier for the maximization problem (2.5), and $\lambda^*_i$ is the Lagrange multiplier for (2.6).

Bowley’s Lemma is frequently used in applied welfare economics and cost-benefit analysis, while the Quadratic Approximation Lemma is frequently used in index number theory, which indicates the close connection between the two fields.

Suppose that we are given a homogeneous translog aggregator function (Christensen, Jorgenson and Lau [1971]) defined by
\[
\ln f(x) = \alpha_0 + \sum_{n=1}^{N} \alpha_n \ln x_n + \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} \gamma_{jk} \ln x_j \ln x_k,
\]
where $\sum_{n=1}^{N} \alpha_n = 1$, $\gamma_{jk} = \gamma_{kj}$ and $\sum_{j=1}^{N} \gamma_{jk} = 0$ for $j = 1, 2, \ldots, N$.

Jorgenson and Lau have shown that the homogeneous translog function can provide a second order approximation to an arbitrary twice continuously differentiable linearly homogeneous function. Let us use the parameters which occur in the translog functional form in order to define the following function, $f^*$:
\[
f^*(z) = \alpha_0 + \sum_{j=1}^{N} \alpha_j z_j + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij} z_i z_j.
\]

Since the function $f^*$ is quadratic, we can apply the Quadratic Approximation Lemma (2.2), and we obtain
\[
f^*(z^1) - f^*(z^0) = \frac{1}{2}[\nabla f^*(z^1) + \nabla f^*(z^0)] \cdot (z^1 - z^0).
\]

Now we relate $f^*$ to the translog function $f$. We have
\[
\nabla f^*(z^r) = \partial f^*(z^r)/\partial z_j = \partial \ln f(x^r)/\partial \ln x_j = [\partial f(x^r)/\partial x_j][x^r_j/f(x^r)],
\]
\[
f^*(z^r) = \ln f(x^r),
\]
\[
z^r_j = \ln x^r_j, \quad \text{for } r = 0, 1 \text{ and } j = 1, 2, \ldots, N.
\]

If we substitute relation (2.10) into (2.9), we obtain
\[
\text{ln } f(x^1) - \ln f(x^0) = \frac{1}{2} \left[ \hat{x}^1 \nabla f(x^1)/f(x^1) + \hat{x}^0 \nabla f(x^0)/f(x^0) \right] \cdot (\ln x^1 - \ln x^0),
\]
where $\ln x^1 = (\ln x^1_1, \ln x^1_2, \ldots, \ln x^1_N)$, $\ln x^0 = (\ln x^0_1, \ln x^0_2, \ldots, \ln x^0_N)$, $\hat{x}^1$ is the vector $x^1$ diagonalized into a matrix, and $\hat{x}^0$ is the vector $x^0$ diagonalized into a matrix.

Assume that $x^r \gg 0_N$ is a solution to the aggregator maximization problem
\[
\text{max}_x \{f(x) : p^r \cdot x = p^r \cdot x^r, \ x \geq 0_N\},
\]
where $p^r \gg 0_N$ for $r = 0, 1$, and $f$ is the homogeneous translog function. The first order conditions for the two maximization problems, after elimination of the Lagrange multipliers (Könüs and Byushgens [1926; 155], Hotelling [1935; 71–74], Wold [1944; 69–71] and Pearce [1964; 59]), yield the relations $p^r/p^r \cdot x^r = \nabla f(x^r)/x^r \cdot \nabla f(x^r)$ for $r = 0, 1$. Since $f$ is linearly homogeneous, $x^r \cdot \nabla f(x^r)$ may be replaced by $f(x^r)$ in the above, and substitution of these last two relations into (2.11) yields
\[
\text{ln } f(x^1)/f(x^0) = \frac{1}{2} \left[ \hat{x}^1 p^1/p^1 x^1 + \hat{x}^0 p^0/p^0 x^0 \right] \cdot (\ln x^1 - \ln x^0)
\]
\[= \sum_{n=1}^{N} \frac{1}{2} (s^1_n + s^0_n) \ln(x^1_n/x^0_n),
\]
or
\[
(2.12) \quad f(x^1)/f(x^0) = \prod_{n=1}^{N} (x_n^1/x_n^0)^{(s_n^1 + s_n^0)/2} \equiv Q_0(p^0, p^1, x^0, x^1),
\]
where \(s_n^r \equiv p_n^r x_n^r / p^r \cdot x^r\), the \(n\)th share of cost in period \(r\).

The right hand side of (2.12) is the quantity index which corresponds to Irving Fisher's [1922] price index number 124, using (1.1). It has also been advocated as a quantity index by Törnqvist [1936] and Theil [1965] [1967] [1968]. It has been utilized empirically by Christensen and Jorgenson [1969] [1970] as a discrete approximation to the Divisia [1926] index and by Star [1974] and Star and Hall [1976] in the context of productivity measurement. The above argument shows that this quantity index is exact for a homogeneous translog aggregator function, and in view of the second order approximation property of the homogeneous translog function, we see that the right hand side of (2.12) is a superlative quantity index.

It can also be seen (using the if and only if nature of the Quadratic Approximation Lemma (2.2)) that the homogeneous translog function is the only differentiable linear homogeneous function which is exact for the Törnqvist–Theil quantity index.

The above argument can be repeated (with some changes in notation) if the unit cost function for the aggregator function is the translog unit cost function defined by
\[
\ln c(p) \equiv \alpha_n^0 + \sum_{j=1}^{N} \alpha_n^* \ln p_n + \frac{1}{2} \sum_{j,k=1}^{N} \gamma_{jk}^r \ln p_j \ln p_k,
\]
where \(\sum_{n=1}^{N} \alpha_n^* = 1\), \(\gamma_{jk}^r = \gamma_{kj}^r\) and \(\sum_{k=1}^{N} \gamma_{jk}^r = 0\) for \(j = 1, 2, \ldots, N\). We also need the following results.

**Lemma 2.13.** (Shephard [1953; 11], Samuelson [1947]): If \(f\) is positive, linearly homogeneous and concave; if
\[
p^r \cdot x^r = \min_x \{p^r \cdot x : f(x) \geq f(x^r)\} = c(p^r)f(x^r) \quad \text{for} \quad r = 0, 1,
\]
and if the unit cost function \(c\) is differentiable at \(p^r\), then
\[
x^r = \nabla c(p^r)f(x^r) \quad \text{for} \quad r = 0, 1.
\]

**Corollary 2.14.**
\[
x^r/p^r \cdot x^r = \nabla c(p^r)/c(p^r) \quad \text{for} \quad r = 0, 1.
\]

Now under the assumption of cost minimizing behavior in periods 0 and 1 (which implies (2.14)), we have upon applying the Quadratic Approximation Lemma (2.2) to the translog unit cost function,
\[
\ln c(p^1) - \ln c(p^0) = \frac{1}{2} \left[ \frac{\partial}{\partial p^1} \ln c(p^1) p^1 \ln c(p^0) + \frac{\partial}{\partial p^0} \ln c(p^0) p^0 \ln c(p^0) \right] \cdot (\ln p^1 - \ln p^0)
\]
\[
= \frac{1}{2} \left[ \frac{\partial}{\partial p^1} \frac{x^1}{p^1 \cdot x^1} + \frac{\partial}{\partial p^0} \frac{x^0}{p^0 \cdot x^0} \right] \cdot (\ln p^1 - \ln p^0) \quad \text{(using (2.14))}
\]
\[
= \sum_{n=1}^{N} (s_n^1 + s_n^0) \ln (p_n^1/p_n^0),
\]
or
\[
(2.15) \quad c(p^1)/c(p^0) = \prod_{n=1}^{N} (p_n^1/p_n^0)^{(s_n^1 + s_n^0)/2},
\]
where \(s_n^r = p_n^r x_n^r / p^r \cdot x^r\) (the \(n\)th share of cost in period \(r\)), \(p^r \gg 0_N\) (period \(r\) prices, \(r = 0, 1\)), \(x^r \geq 0_N\) (period \(i\) quantities, \(i = 0, 1\)), and \(c(p) = \ln c(p)\) is the translog unit cost function.

The right hand side of (2.15) corresponds to Irving Fisher's [1922] price index 123. The above argument shows that this price index is exact for a translog unit cost function, and that this is the only differentiable unit cost function which is exact for this price index.

Let us denote the right hand side of (2.15) as the price index function \(P_0(p^0, p^1, x^0, x^1)\), and denote the right hand side of (2.12) as the quantity index \(Q_0(p^0, p^1, x^0, x^1)\). It can be verified that \(P_0(p^0, p^1, x^0, x^1) = Q_0(p^0, p^1, x^0, x^1)\) if the price index \(P_0\) and the quantity index \(Q_0\) do not satisfy the weak factor reversal test (1.1). This is perfectly reasonable, since the quantity index \(Q_0\) is consistent with a homogeneous translog (direct) aggregator function, while the price index \(P_0\) is consistent with an aggregator function which is dual to the translog unit cost function, and the two aggregator functions do not in general coincide; i.e., they correspond to different (aggregation) technologies. Thus, given \(Q_0\), the corresponding price index, which satisfies (1.1), is defined by \(P_0(p^0, p^1, x^0, x^1) = p^1 \cdot x^1 / [p^0 \cdot x^0 Q_0(p^0, p^1, x^0, x^1)]\). The quantity index \(Q_0\) and the corresponding (implicit) price index \(P_0\) were used by Christensen and Jorgenson [1969] [1970].

1 **Proof:** divide the equation in Lemma 2.13 by \(p^r \cdot x^r = c(p^r)f(x^r)\).

2 **Note:** the validity of (2.15) depends crucially on the validity of (2.14), which will be valid if \(p^0\) and \(p^1\) belong to an open convex set of prices \(P\), such that the translog \(c(p)\) satisfies the regularity conditions of positivity, linear homogeneity and concavity over \(P\).
in order to measure U.S. real input and output. On the other hand, given \( P_0 \), the corresponding (implicit) quantity index, which satisfies (1.1), is defined by

\[
Q_0(p^0, p^1, x^0, x^1) = p^1 \cdot x^1 / [p^0 \cdot x^0 P_0(p^0, p^1, x^0, x^1)].
\]

The price-quantity index pair \((P_0, Q_0)\) was advocated by Kloeck [1967: 2] over the pair \((\tilde{P}_0, \tilde{Q}_0)\) on the following grounds: as we disaggregate more and more, we can expect the individual consumer or producer to utilize positive amounts of fewer and fewer goods (i.e., as \( N \) grows, components of the vectors \( x^0 \) and \( x^1 \) tend to become zero), but the prices which the producer or consumer faces are generally positive irrespective of the degree of disaggregation. Since the logarithm of zero is not finite, \( Q_0 \) will tend to be indeterminate as the degree of disaggregation increases, but \( P_0 \) will still be well defined (provided that all prices are positive).

Theil [1968] and Kloeck [1967] provided a somewhat different interpretation of the indexes \( P_0 \) and \( Q_0 \), an interpretation which does not require the aggregator function to be linear homogeneous. Let the aggregate \( u \) be defined by \( u = f(x) \), where \( f \) is a not necessarily homogeneous aggregator function which satisfies for example the Shephard [1970] or Diewert [1971a] regularity conditions for a production function. For \( p \gg 0, Y > 0 \), define the total cost function by \( C(u, p) \equiv \min_x \{ p \cdot x : f(x) \geq u; x \geq 0 \} \) and the indirect utility function by \( g(p/Y) \equiv \max_x \{ f(x) : p \cdot x \leq Y, x \leq 0 \} \). The true cost of living price index evaluated at ‘utility’ level \( u \) is defined as \( P(p^0, p^1, u) \equiv C(u^1, p^1)/C(u^0, p^0) \) and the Theil index of quantity (or ‘real income’) evaluated at prices \( p \) is defined as \( Q_T(p; u^0, u^1) \equiv C(u^1, p)/C(u^0, p) \). The Theil-Kloeck results are that:

(i) \( P_0(p^0, p^1, x^0, x^1) \) is a second order approximation to \( P(p^0, p^1, g(u^*)) \), where the \( n \)th component of \( u^* \) is \( \alpha_n^* \equiv (\gamma_n^*, p_n^0, \beta_n^*x^0, x^1)^{1/2} \), for \( n = 1, 2, \ldots, N \), and

(ii) \( Q_0(p^0, p^1, x^0, x^1) \) is a second order approximation to \( Q_T[p^*, g(p^0, p^1, x^0, x^1), g(p^1, p^0, x^0, x^1)] \), where the \( n \)th component of \( p^* \) is \( p_n^* \equiv (\gamma_n^*, p_n^0)^{1/2} \).

In view of the Theil–Kloeck approximation results, we might be led to ask whether the index number \( P_0 \) is exact for any general (nonhomothetic) functional forms for the cost function \( C(u, p) \). The following theorem answers this question in the affirmative:

**Theorem 2.16.** Let the functional form for the cost function be a general translog of the form

\[
\ln C(u, p) = \alpha_u^* + \sum_{i=1}^{N} \alpha_i^* \ln p_i + \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} \gamma_{jk}^* \ln p_j \ln p_k
\]

\[
+ \beta^* \ln u + \frac{1}{2} \delta^*(\ln u)^2 + \sum_{k=1}^{N} \epsilon_k^* \ln u \ln p_i
\]

where \( \sum_{i=1}^{N} \alpha_i^* = 1, \quad \gamma_{jk}^* = \gamma_{kj}^*, \quad \sum_{k=1}^{N} \gamma_{jk}^* = 0, \quad \text{for} \quad j = 1, 2, \ldots, N, \quad \text{and} \quad \sum_{i=1}^{N} \epsilon_i^* = 0.3 \)

Let \((u^0, p^0)\) and \((u^1, p^1)\) belong to a \((u, p)\) region where \( C(u, p) \) satisfies the appropriate regularity conditions for a cost function (e.g., see Shephard [1970], Hanoch [1970] or Diewert [1971a]) and define the quantity vectors \( x^0 \equiv \nabla_u C(u^0, p^0) \) and \( x^1 \equiv \nabla_p C(u^1, p^1) \). Then

\[
P_0(p^0, p^1, x^0, x^1) = C(u^*, p^1)/C(u^*, p^0),
\]

where \( u^* \equiv (u^0 u^1)^{1/2} \) and \( P_0 \) is defined by the right hand side of (2.15).

In contrast to the case of a linearly homogeneous aggregator function where the cost function takes the simple form \( C(u, p) = c(p)u \), Theorem 2.16 is not an if and only if result; that is, the index number \( P_0(p^0, p^1, x^0, x^1) \) is exact for functional forms for \( C(u, p) \) other than the translog. In fact, Theorem 2.16 remains true if: (i) we define \( C \) as \( \ln C(u, p) = \alpha_0(u) + \sum_{i=1}^{N} \{ \alpha_i + \epsilon_i h(u) \} \ln p_i + \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} \gamma_{jk} \ln p_j \ln p_k \), where \( \sum_{i=1}^{N} \alpha_i = 1, \quad \sum_{i=1}^{N} \epsilon_i = 0, \quad \gamma_{jk} = \gamma_{kj} \), \( \sum_{k=1}^{N} \gamma_{jk} = 0 \), for \( j = 1, 2, \ldots, N, \) and \( h \) is a monotonically increasing function of one variable, and (ii) define the reference utility level \( u^* \) as the solution to the equation \( 2h(u^*) = h(u^1) + h(u^0) \). (In the translog case, \( h(u) \equiv \ln u \).)

Thus the same price index \( P_0 \) is exact for more than one functional form (and reference utility level) for the true cost of living.

We can also provide a justification for the quantity index \( Q_0 \) in the context of an aggregator function \( f \) which is not necessarily linearly homogeneous. In order to provide this justification, it is necessary to define the quantity index which has been proposed by Malmquist [1953] and Pollak [1971a] in the context of consumer theory, and by Bergson [1961] and Moorsteen [1961] in the context of producer theory.

Given an aggregator function \( f \) and an aggregate \( u := f(x) \), define \( f \)’s distance function as \( D(u, x) = \max_k \{ k : f(x/k) \geq u \} \). To use the language of utility theory, the distance function tells us by what proportion one has to deflate (or inflate) the given consumption vector \( x \) in order to obtain a point on the utility surface indexed by \( u \). It can be shown that if \( f \) satisfies certain regularity conditions, then \( f \) is completely characterized by \( D \) (see Shephard [1970], Hanoch [1970] and McFadden [1970]). In particular, \( D(u, x) \) is linearly homogeneous, nondecreasing and concave in the vector of variables \( x \) and nonincreasing in \( u \) in Hanoch’s formulation.

Now define the Malmquist quantity index as \( Q_M(x^0, x^1, u) \equiv D(u, x^1)/D(u, x^0) \). Note that the index depends on \( x^0 \) (the base period quantities), \( x^1 \) (the current period quantities) and on the base indifference surface (which is indexed by \( u \)) onto which the points \( x^0 \) and \( x^1 \) are deflated. The following theorem relates the translog functional form to the Malmquist quantity index.

**Theorem 2.17.** Let an aggregator function \( f \) satisfying the Hanoch [1970] and Diewert [1971a] regularity conditions be given such that \( f \)’s distance function \( D \)
is a general translog of the form
\[
\ln D(u, x) = \alpha_0 + \sum_{i=1}^{N} \alpha_i \ln x_i + \frac{1}{2} \sum_{k=1}^{N} \sum_{j=1}^{N} \gamma_{jk} \ln x_j + \beta \ln u + \frac{1}{2} \delta (\ln u)^2 + \sum_{i=1}^{N} \varepsilon_i \ln u \ln x_i,
\]
where \(\sum_{i=1}^{N} \alpha_i = 1, \gamma_{jk} = \gamma_{kj}, \sum_{k=1}^{N} \gamma_{jk} = 0, \) for \(j = 1, 2, \ldots, N, \) and \(\sum_{i=1}^{N} \varepsilon_i = 0.\) Suppose that the quantity vector \(x^0\) is a solution to the aggregator maximization problem \(\max_x \{f(x) : p^0 \cdot x = p^0 \cdot x^0\},\) while \(x^1\) is a solution to \(\max_x \{f(x) : p^1 \cdot x = p^1 \cdot x^1\}\) and \(u^0 = f(x^0), u^1 = f(x^1)\). Then
\[
Q_0(p^0, p^1, x^0, x^1) = D(u^*, x^1) / D(u^*, x^0) \equiv Q_M(x^0, x^1, u^*),
\]
where \(u^* \equiv (u^0 u^1)^{1/2}\) and \(Q_0\) is defined in (2.12).

As was the case with the price index \(P_0,\) the quantity index \(Q_0\) is equal to Malmquist quantity indexes which are defined by nontranslog distance functions; i.e., Theorem (2.17) is not an if and only if result. However, Theorems (2.16) and (2.17) do provide a rather strong justification for the use of \(P_0\) or \(Q_0\) since the translog functional form provides a second order approximation to a general cost or distance function (which in turn are dual to a general nonhomothetic aggregator function).

Finally, note that Theorems (2.16) and (2.17) have a ‘global’ character to them in contrast to the Theil–Kloek ‘local’ results.

3. Productivity Measurement and ‘Divisia’ Indexes

Jorgenson and Griliches [1972; 83–84] have advocated the use of the indexes \(\overline{P}_0, Q_0\) in the context of productivity measurement. It is perhaps appropriate to review their procedure in the light of the results of the previous section.

First, we note (by a straightforward computation) that it is not in general true that ‘a discrete Divisia index of discrete Divisia indexes is a discrete Divisia index of the components’ (Jorgenson and Griliches [1972; 83]), where the ‘Divisia’ quantity index is defined to be \(Q_0\). In view of the one-to-one nature of the index number \(Q_0\) with the translog functional form for the aggregator function \(f\) in the linearly homogeneous case, it can be seen that the Jorgenson–Griliches assertion will be true if the producer or consumer is maximizing an aggregator function \(f\) subject to an expenditure constraint, where \(f\) is both a homogeneous translog function and a translog of micro translog aggregator functions. The set of such translog functions is not empty since it contains the set of Cobb–Douglas functions. Thus if cost shares are approximately constant (which corresponds to the Cobb–Douglas case), then the Jorgenson–Griliches assertion will be approximately true.

It can be similarly shown that, in general, it is not true that a discrete ‘Divisia’ price index of discrete ‘Divisia’ indexes is a discrete ‘Divisia’ price index of the components, where the ‘Divisia’ price index is defined to be \(P_0:\) the first method of constructing a price index is justified if the aggregator function has a unit cost function dual of the form \(\hat{c} [\lambda^1(p^1), \lambda^2(p^2), \ldots, \lambda^N(p^N)],\) where \((p^1, p^2, \ldots, p^N)\) represents a partition of the price vector \(p\) and the functions \(\hat{c}, c^1, c^2, \ldots, c^N\) are all translog unit cost functions, while the second method of constructing a price index is justified if the aggregator function has a unit cost function dual, \(c(p),\) which is translog.

Jorgenson and Griliches [1972] use the index number formula \(Q_0(p^0, p^1, x^0, x^1)\) defined by the right hand side of (2.12) not only to form an index of real input, but also to form an index of real output. Just as the aggregation of inputs into a composite input rests on the duality between unit cost and homogeneous production functions, the aggregation of outputs into a composite output can be based on the duality between unit revenue and homogeneous factor requirements functions.\(^4\) We briefly outline this latter duality.

Suppose that a producer is producing \(M\) outputs, \((y_1, y_2, \ldots, y_M) \equiv y,\) and the technology of the producer can be described by a factor requirements function, \(g,\) where \(g(y) = \) the minimum amount of aggregate input required to produce the vector of outputs \(y.\)\(^5\) The producer’s unit (aggregate input) revenue function\(^6\) is defined for each price vector \(w \geq 0_M\) by
\[
r(w) \equiv \max_y \{w \cdot y : g(y) \leq 1, y \geq 0_M\}.
\]

Thus given a factor requirements function \(g,\) (3.1) may be used to define a unit revenue function. On the other hand, given a unit revenue function \(r(w)\) which is a positive, linearly homogeneous, convex function for \(w \geq 0_M,\) a factor requirements functions \(g^*\) consistent with \(r\) may be defined for \(y \geq 0_M\)


\(^4\)Assume \(g\) is defined for \(y \geq 0_M,\) and has the following properties: (i) \(g(y) > 0\) for \(y \gg 0_M\) (positivity), (ii) \(g(\lambda y) = \lambda g(y)\) for \(\lambda \geq 0, y \geq 0\) (linear homogeneity), and (iii) \(g(\lambda y^1 + (1-\lambda) y^2) \leq \lambda g(y^1) + (1-\lambda) g(y^2)\) for \(0 \leq \lambda \leq 1, y^1 \geq 0_M, y^2 \geq 0_M\) (convexity).

\(^5\)If \(g\) satisfies the three properties listed in footnote 5, then \(r\) also has those three properties.
by\footnote{The proof is analogous to the proof of the Samuelson–Shephard duality theorem presented in Dievert [1974a]; alternatively, see Samuelson and Swamy [1974].}

\begin{equation}
(3.2) \quad g^r(y) \equiv \min_{\lambda} \{ \lambda : w \cdot y \leq r(w) \lambda \text{ for every } w \geq 0_M \}
\end{equation}

\[ = \min_{\lambda} \{ \lambda : 1 \leq r(w) \lambda \text{ for every } w \geq 0_M \text{ such that } w \cdot y = 1 \}
\]

\[ = 1/ \min_w \{ r(w) : w \cdot y = 1, w \geq 0_M \}. \]

The translog functional form may be used to provide a second order approximation to an arbitrary twice differentiable factor requirements function. Thus assume that \( g \) is defined (at least over the relevant range of \( y \)'s) by

\begin{equation}
(3.3) \quad \ln g(y^r) \equiv a_0 + \sum_{m=1}^{M} a_m \ln y_m^r + \frac{1}{2} \sum_{j=1}^{M} \sum_{k=1}^{M} c_{jk} \ln y_j^r \ln y_k^r, \quad \text{for } r = 0, 1,
\end{equation}

where

\[ \sum_{m=1}^{M} a_m = 1, \quad c_{jk} = c_{kj}, \quad \sum_{k=1}^{M} c_{jk} = 0, \quad \text{for } j = 1, 2, \ldots, M. \]

Now assume that \( y^r \gg 0_M \) is a solution to the aggregate input minimization problem \( \min_y \{ g(y) : w^r \cdot y = w^r \cdot y^r, \ y \geq 0_M \} \), where \( w^r \gg 0_M \) for \( r = 0, 1 \), and \( g \) is the translog function defined by (3.3). Then the first order necessary conditions for the minimization problems along with the linear homogeneity of \( g \) yield the relations \( w^r / w^r \cdot y^r = \nabla g(y^r)/g(y^r) \), for \( r = 0, 1 \), and using these two relations in Lemma (2.2) applied to (3.3), we deduce that

\begin{equation}
(3.4) \quad g(y^1)/g(y^0) = \prod_{m=1}^{M} \left( \frac{y_m^1}{y_m^0} \right)^{(w_m^1/y_m^0)(w^0_m/y_m^1)+(w_m^0/y_m^1)/2}
\equiv Q_0^r(w^0, w^1; y^0, y^1).
\end{equation}

Thus again the Törnqvist formula can be used to aggregate quantities consistently, provided that the underlying aggregator function is a homogeneous translog.

Similarly if the revenue function \( r(w) \) is translog over the relevant range of data and if the producer is in fact maximizing revenue, then we can show that \( r(w^1)/r(w^0) = P^r(w^0, w^1, y^0, y^1) \equiv Q_0^r(y^0, y^1, w^0, w^1) \), the Törnqvist price index. (Note that in \( Q_0^r(y^0, y^1, w^0, w^1) \), prices and quantities are interchanged compared to \( Q_0^r(w^0, w^1, y^0, y^1) \) which appeared in (3.4) above.)

Using the above material, we may now justify the Jorgenson–Griliches [1972] method of measuring technical progress. Assume that the production possibilities efficient set can be represented as a set of outputs \( y \) and inputs \( x \) such that

\begin{equation}
(3.5) \quad g(y) = f(x),
\end{equation}

where \( g \) is the homogeneous translog factor requirements function defined by (3.3), and \( f \) is the homogeneous translog production function defined in Section 2. Let \( w^r \gg 0_M, p^r \gg 0_N, r = 0, 1 \) be vectors of output and input prices during periods 0 and 1, and assume that \( y^0 \gg 0_M \) and \( x^0 \gg 0_N \) is a solution to the period 0 profit maximization problem,

\begin{equation}
(3.6) \quad \max_{y, x} \{ w^0 \cdot y - p^0 \cdot x : g(y) = f(x) \}.
\end{equation}

Suppose that ‘technical progress’ occurs between periods 0 and 1 which we assume to be a parallel outward shift of the ‘isoquants’ of the aggregator function \( f \); i.e., we assume that the equation which defines the efficient set of outputs and inputs in period 1 is \( g(y) = (1 + \tau)f(x) \) where \( \tau \) represents the amount of ‘technical progress’ if \( \tau > 0 \) or ‘technical regress’ if \( \tau < 0 \). Finally, assume that \( y^1 \gg 0_M \) and \( x^1 \gg 0_N \) is a solution to the period 1 profit maximization problem,

\begin{equation}
(3.7) \quad \max_{y, x} \{ w^1 \cdot y - p^1 \cdot x : g(y) = (1 + \tau)f(x) \}.
\end{equation}

Thus we have \( g(y^0) = f(x^0) \) and \( g(y^1) = (1 + \tau)f(x^1) \). It is easy to see that \( y^r \gg 0_M \) is a solution to the aggregate input minimization problem \( \min_y \{ g(y) : w^r \cdot y = w^r \cdot y^r, \ y \geq 0_M \} \), for \( r = 0, 1 \), and thus (3.4) holds. Similarly, \( x^r \gg 0_N \) is a solution to the aggregator maximization problem \( \max_x \{ f(x) : p^r \cdot x = p^r \cdot x^r, \ x \geq 0_N \} \), for \( r = 0, 1 \), and thus (2.12) holds. Substitution of (2.12) and (3.4) into the identity \( g(y^1)/g(y^0) = (1 + \tau)f(x^1)/f(x^0) \) yields the following expression for \( (1 + \tau) \) in terms of observable prices and quantities:

\begin{equation}
(3.8) \quad (1 + \tau) = \frac{\prod_{m=1}^{M} \left( y_m^1/y_m^0 \right)^{(w_m^1/y_m^0)(w_m^0/y_m^1)+(w_m^0/y_m^1)/2}}{\prod_{n=1}^{N} \left( x_n^1/x_n^0 \right)^{(p_n^1/x_n^0)(p_n^0/x_n^1)+(p_n^0/x_n^0)/2}}.
\end{equation}

Thus the Jorgenson–Griliches method of measuring technical progress can be justified if: (i) the economy’s production possibilities set can be represented by a separable transformation surface defined by \( g(y) = f(x) \), where
the input aggregator function \( f \) and the output aggregator function \( g \) are both homogeneous translog functions, (ii) producers are maximizing profits, and (iii) technical progress takes place in the ‘neutral’ manner postulated above.

Since the separability assumption \( g(y) = f(x) \) is somewhat restrictive from an a priori point of view, it would be of some theoretical point of interest to devise a measure of technical progress which did not depend on this separability assumption. This can be done, but only at a cost as we shall see below.

Suppose that technology can be represented by a transformation function,\(^8\) where \( y_1 = t(y_2, y_3, \ldots, y_M; x_1, x_2, \ldots, x_N) \equiv t(\tilde{y}; x) \equiv t(z) \) is the maximum amount of output one that can be produced given that the vector of other outputs \( \tilde{y} = (y_2, y_3, \ldots, y_M) \) is to be produced by the vector of inputs \( x = (x_1, x_2, \ldots, x_N) \). Assume that \( t \) is a positive, linearly homogeneous, concave function over a convex set of the nonnegative orthant \( S \) in \( K = M-1+N \) dimensional space. Assume also that \( t(\tilde{y}; x) \) is nonincreasing in the components of the other outputs vector \( \tilde{y} \) and nondecreasing in the components of the input vector \( x \). Suppose that the transformation function \( t \) is defined for \( z \) belonging to \( S \) by

\[
(3.9) \quad \ln t(z) = \alpha_0 + \sum_{k=1}^{K} \ln z_k + \frac{1}{2} \sum_{j=1}^{K} \gamma_{jk} \ln z_j \ln z_k,
\]

where \( \sum_{k=1}^{K} \alpha_k = 1, \gamma_{jk} = \gamma_{kj} \) and \( \sum_{k=1}^{K} \gamma_{jk} = 0 \), for \( j = 1, 2, \ldots, K; \) i.e., \( t \) is a translog transformation function over the set \( S \).

Suppose that \( y' = (y'_1, y'_2, \ldots, y'_M) \gg 0_M \) (output vectors), \( x' = (x'_1, x'_2, \ldots, x'_N) \gg 0_N \) (input vectors), \( w' \gg 0_M \) (input price vectors), and \( u' \gg 0_N \) (output price vectors), and \( w' \cdot y' = p' \cdot x' \) (value of outputs equals value of inputs) for periods \( r = 0, 1 \). Assume that \( z^0 = (y^0_1, y^0_2, \ldots, y^0_m, x^0_1, \ldots, x^0_N) \equiv (\tilde{y}^0, x^0) \) is a solution to the following output maximization subject to an expenditure constraint problem in period 0:

\[
(3.10) \quad \max_z \{t(z) : q^0 \cdot z = q^0 \cdot z^0, \quad z \text{ belongs to } S\}
\]

where \( t \) is the translog transformation function defined by \( (3.9) \), \( q^0 \equiv (-w^0, -w^0, \ldots, -w^0_M; p^0_1, p^0_2, \ldots, p^0_N) \equiv (\tilde{w}^0; p^0) \), and

\[
(3.11) \quad y^0 = t(z^0).\(^9\)
\]

\(^8\)For a more detailed discussion of transformation functions and their properties, see Diewert [1973a].

\(^9\)Note that \( q^0 \cdot z^0 = w^0_1 \cdot y^0 > 0 \), since \( w^0 \cdot y^0 = p^0 \cdot x^0 \).

The first order conditions for the maximization problem \( (3.10) \) imply that (Konüs–Byushgens [1926] Lemma)

\[
(3.12) \quad q^0 / q^0 \cdot z^0 = \nabla t(z^0) / t(z^0).
\]

As before, we assume that ‘neutral’ input augmenting technical progress takes place between periods 0 and 1; i.e., if \( (y; x) \) was an efficient vector of outputs and inputs in period 0, then \( [y; (1 + \tau)^{-1} x] \) is on the efficiency surface in period 1. Thus the efficiency surface in period 1 can be defined as the set of \( (y_1, y_2, \ldots, y_M; x_1, x_2, \ldots, x_N) \) which satisfy the following equation:

\[
(3.13) \quad y_1 = t(y_2, y_3, \ldots, y_M; (1 + \tau)x_1, (1 + \tau)x_2, \ldots, (1 + \tau)x_N).
\]

Assume that \( (y'_1, y'_2, \ldots, y'_M; x'_1, x'_2, \ldots, x'_N) \equiv (\tilde{y}^1; x^1) \) is a solution to the period 1 output maximization subject to an expenditure constraint problem \( \max_{y,x} \{t(\tilde{y}; (1 + \tau)x) : -\tilde{w}^1 \cdot y + p^1 \cdot x = -\tilde{w}^1 \cdot y^1 + p^1 \cdot x^1; \tilde{y} : (1 + \tau)x \} \) belongs to \( S \). Then \( z^1 \equiv [\tilde{y}^1; (1 + \tau)x^1] \) is a solution to the following output maximization problem:

\[
(3.14) \quad \max_z \{t(z) : q^1 \cdot z = q^1 \cdot z^1, \quad z \text{ belongs to } S\}
\]

where \( t \) is the translog function defined by \( (3.9) \), \( q^1 \equiv (-\tilde{w}^1; p^1) \), and

\[
(3.15) \quad y^1 = t(z^1) = t(\tilde{y}^1; (1 + \tau)x^1).
\]

Again, the Konüs-Byushgens-Hotelling Lemma applied to the maximization problem \( (3.14) \), using the linear homogeneity of \( t \), implies that\(^10\)

\[
(3.16) \quad q^1 / q^1 \cdot z^1 = \nabla t(z^1) / t(z^1).
\]

Now substitute \( (3.12) \) and \( (3.16) \) into the identity \( (2.11) \), except that \( t \) replaces \( f \) and \( z \) replaces \( x \), and we obtain

\[
(3.17) \quad t(z^1) / t(z^0) = \sum_{k=1}^{K} (z^1_k/z^0_k)(q^1_k z_k/q^1 \cdot z^1) + (q^0_k z^0_k/q^0 \cdot z^0) / 2
\]

\[
= t(\tilde{y}^1; (1 + \tau)x^1) / t(y^0; x^0).
\]

Combining \( (3.13), (3.15) \) and \( (3.17) \), we obtain the following equation in \( \tau \):

\[
(3.18) \quad y^1 / y^0 = \prod_{m=1}^{M} (y^1_m / y^0_m)[w^1_m y^1_m / V^1(\tau) + w^0_m y^0_m / V^0(\tau)] / 2
\]

\[
/ \prod_{m=2}^{M} (y^1_m / y^0_m)[w^1_m y^1_m / V^1(\tau) + w^0_m y^0_m / V^0(\tau)] / 2.
\]

\(^10\)We assume that \( \tau \) is small so that \( q^1 \cdot z^1 \equiv -\tilde{w}^1 \cdot \tilde{y}^1 + p^1 \cdot (1 + \tau)x^1 > 0 \).
where $V^0 = - \sum_{m=2}^{M} u_m^0 y_m^0 + \sum_{n=1}^{N} p_n^0 x_n^0$ is the net cost of producing output $y_1$ in period 0, and $V^1(\tau) = - \sum_{m=2}^{M} u_m^1 y_m^1 + \sum_{n=1}^{N} p_n^1 (1 + \tau) x_n^1$.

Given data on outputs, inputs and prices, equation (3.18) can be solved for the unknown rate of technical progress $\tau$. Note that equation (3.18) is quite different from the Jorgenson–Griliches equation for $\tau$ defined by (3.8) (except that the two equations are equivalent when $M = 1$; i.e., when there is only one output).

However, it should be pointed out that our more general measure of technical progress, which is obtained by solving (3.18) for $\tau$, suffers from some disadvantages: (i) our procedure is computationally more difficult, and (ii) our procedure is not symmetric in the outputs (that is, the first output $y_1$ is asymmetrically singled out in (3.18)). Thus different orderings of the outputs could give rise to different measures of technical progress. This is because each ordering of the outputs corresponds to a different translog assumption about the underlying technology and thus different measures of $\tau$ can be obtained. However, all of these measures should be close in empirical applications since the different translog functions are all approximating the same technology to the second order.

4. Quadratic Means of Order $r$ and Exact Index Numbers

For $r \neq 0$, the (homogeneous) quadratic mean of order $r$ aggregator function is defined by

$$f_r(x) = \left[ \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} x_i^{r/2} x_j^{r/2} \right]^{1/r},$$

where $a_{ij} = a_{ji}$, $1 \leq i, j \leq N$, are parameters, and the domain of definition of $f_r$ is restricted to $x \equiv (x_1, x_2, \ldots, x_N) \geq 0$ such that $\sum \sum a_{ij} x_i^{r/2} x_j^{r/2} > 0$, and $f_r$ is concave. The above functional form is due to McCarthy [1967], Kadiyala [1971–72], Denny [1972] [1974] and Hasenkamp [1973]. Denny also defined the quadratic mean of order $r$ unit cost function,

$$c_r(p) \equiv \left[ \sum_{i=1}^{N} \sum_{j=1}^{N} b_{ij} p_i^{r/2} p_j^{r/2} \right]^{1/r}, \quad b_{ij} = b_{ji}, \quad r \neq 0.$$

11Furthermore, we cannot a priori rule out the possibility that equation (3.18) will have either multiple solutions for $\tau$ or no solutions at all. The Fisher measure of technical progress, to be introduced in Section 5, overcomes these difficulties.

12Denny noted that if $r = 1$, then (4.1) reduces to the generalized linear functional form (Dievert [1969] [1971a]), (4.2) reduces to the generalized Leontief functional form (Dievert [1969] [1971a]), and if all $a_{ij} = 0$ for $i \neq j$, then (4.1) reduces to the CES functional form (Arrow, Chenery, Minhas and Solow [1961]), while if all $b_{ij} = 0$ for $i \neq j$, then (4.2) reduces to the CES unit cost function.

We may also note that when $r = 2$, (4.1) reduces to the Konüs–Byushgens [1926] homogeneous quadratic production or utility function, while (4.2) reduces to the Konüs–Byushgens unit cost function. This functional form has also been considered by Afriat [1972b] [1972] and Pollak [1971a] in the context of utility functions and by Dievert [1969] [1974b] in the context of revenue and factor requirements functions.

Lau [1973] has shown that the limit as $r$ tends to zero of the quadratic mean of order $r$ aggregator function (4.1) is the homogeneous translog aggregator function and similarly that the limit as $r$ tends to zero of (4.2) is the translog unit cost function.

This completes our discussion of special cases of the above family of functional forms. The following theorem shows that the functional form is ‘flexible’.

Theorem 4.3. Let $f$ be any linearly homogeneous, twice continuously differentiable positive function defined over an open subset of the positive orthant in $N$ dimensional space. Then for any $r \neq 0$, $f_r$ defined by (4.1) can provide a second order differential approximation to $f$.

By a second order differential approximation to $f$ at a point $x^* \gg 0_N$, (4.1) we mean that there exists a set of $a_{ij}$ parameters for $f_r$ defined by (4.1), such that $f_r(x^*) = f(x^*)$, $\nabla f_r(x^*) = \nabla f(x^*)$, and $\nabla^2 f_r(x^*) = \nabla^2 f(x^*)$; i.e., the values of $f_r$ and $f$ and their first and second order partial derivatives at $x^*$ all coincide.

Define the quadratic mean of order $r$ quantity index $Q_r$ for $x^0 \gg 0_N$, $x^1 \gg 0_N$, $p^0 > 0_N$, $p^1 > 0_N$, for $r \geq 0$, as

$$Q_r(p^0, p^1; x^0, x^1) = \left[ \frac{\sum_{i=1}^{N} (x_i^1/x_i^0)^{r/2} (p_i^0/x_i^0)}{\sum_{k=1}^{N} (x_k^0/x_k^1)^{r/2} (p_k^1/x_k^1)} \right]^{1/r}$$

$$= \left[ \sum_{i=1}^{N} (x_i^1/x_i^0)^{r/2} s_i^0 \right]^{1/r} \left[ \sum_{k=1}^{N} (x_k^0/x_k^0)^{-r/2} s_k^1 \right]^{-1/r}.$$

Thus for any $r \neq 0$, $Q_r$ may be calculated as a function of observable prices and quantities in two periods. Note that $Q_r$ can be expressed as the
product of a mean of order \(r^{13}\) in the square roots of the quantity relatives \((x_i^0/x_0^0)^{1/2}\) (using base period cost shares as weights) times a mean of order \(-r\) in the square roots of the quantity relatives \((x_i^0/x_0^0)^{1/2}\) (using period one cost shares as weights).

It is perhaps of some interest to note which of Irving Fisher’s [1911] [1922] tests are satisfied by the quantity index \(Q_r\). It can be verified that \(Q_r\) satisfies: (i) the commodity reversal test (i.e., the value of the index number does not change if the ordering of the commodities is changed); (ii) the identity test (i.e., \(Q_r(p_0, p_i; x_0, x^0) \equiv 1\)); (in fact \(Q_r(p_0, p_i; x_0, x^0) \equiv 1\) with the quantity index equal to one so long as all quantities remain unchanged); (iii) the commensurability test (i.e., \(Q_r(D^{-1}p_0, D^{-1}p_i, Dx_0, Dx^0) = Q_r(p_0, p_i, x_0, x^0)\) where \(D\) is a diagonal matrix with positive elements down the main diagonal so that the quantity index remains invariant to changes in units of measurement); (iv) the determinateness test (i.e., \(Q_r(p_0, p_i, x_0, x^1)\) does not become zero, infinite or indeterminate if an individual price becomes zero for any \(r \neq 0\) and \(Q_r(p_0, p_i; x_0, x^1)\) does not become zero, infinite or indeterminate if an individual quantity becomes zero if \(0 < r \leq 2\));\(^{14}\) (v) the proportionality test (i.e., \(Q_r(p^0, p_i; x_0^0, \lambda x^0) = \lambda\) for every \(\lambda > 0\); and (vi) the time or point reversal test (i.e., \(Q_r(p_0, p_i; x_0^0, x^1)Q_r(p_0, p_i; x_0, x^0) \equiv 1\).

Define the quadratic mean of order \(r\) price index \(P_r\) for \(p^0 \gg 0_N, p_i \gg 0_N, x^0 > 0_N, x^1 > 0_N\), for \(r \neq 0\), as

\[
P_r(p_0, p_i; x_0^0, x^1) \equiv \left[ \frac{\sum_{k=1}^{N} (p_k^0/p_k^0)^{r/2} (x_k^0/x_k^0)^{r/2}}{\sum_{k=1}^{N} (p_k^0/p_k^0)^{r/2} (x_k^0/x_k^0)^{r/2}} \right]^{1/r} = Q_r(x_0^0, x^1; p_0^0, p_i^0).
\]

It is easy to see that \(P_r\) will also satisfy Fisher’s tests (i) to (vi). The only Fisher tests not satisfied by the indexes \(P_r\) and \(Q_r\) are: (vii) the circularity test (i.e., \(P_r(p_0^0, p_i^0; x_0^0, x^0)P_r(p_0^0, p_i^0; x_1^0, x^0) \neq P_r(p_0^0, p_i^0; x_0^0, x^0)\), and (viii) the factor reversal test (i.e., \(P_r(p_0^0, p_i^0; x_0^0, x^1)Q_r(p_0^0, p_i^0; x_0^0, x^1) \neq p_0^1 \cdot x_0^1/p_0^0 \cdot x_0^1\) except that \(P_2\) and \(Q_2\), the ‘ideal’ price and quantity indexes, satisfy the factor reversal test).

For \(r \neq 0\) define the implicit quadratic mean of order \(r\) price index \(\tilde{P}_r\) as

\[
\tilde{P}_r(p_0^0, p_i^0; x_0^0, x^1) \equiv p_1^1 \cdot x_1^1 / [p_0^0 \cdot x_0^0 Q_r(p_0^0, p_i^0; x_0^0, x^0)],
\]

and define the implicit quadratic mean of order \(r\) quantity index \(\tilde{Q}_r\) as

\[
\tilde{Q}_r(p_0^0, p_i^0; x_0^0, x^1) \equiv p_1^1 \cdot x_1^1 / [p_0^0 \cdot x_0^0 P_r(p_0^0, p_i^0; x_0^0, x^0)].
\]

\(^{13}\)Ordinary, as opposed to quadratic, means of order \(r\) were defined by Hardy, Littlewood and Polya [1934].

\(^{14}\)Thus the quantity indexes \(Q_r\), for \(0 < r \leq 2\), are somewhat more satisfactory than the Törnqvist–Theil index \(Q_0\) defined by (2.12).

Thus the two pairs of indexes \((Q_r, \tilde{P}_r)\) and \((\tilde{Q}_r, P_r)\) will satisfy the weak factor reversal test (1.1).

The following theorem relates the aggregator function \(f_r\) to the quantity index \(Q_r\):\

**Theorem 4.8.** Suppose that (i) \(f_r(x)\) is defined by (4.1), where \(r \neq 0\); (ii) \(x^0 \gg 0_N\) is a solution to the maximization problem \(\max_x \{f_r(x) : p^0 \cdot x \leq p^0 \cdot x^0, x \text{ belongs to } S\}\), where \(S\) is a convex subset of the nonnegative orthant in \(\mathbb{R}^N\), \(f_r(x^0) > 0\) and the price vector \(p^0\) is such that \(p^0 \cdot x^0 > 0\); and (iii) \(x^1 \gg 0_N\) is a solution to the maximization problem \(\max_x \{f_r(x) : p^0 \cdot x \leq p^1 \cdot x, x \text{ belongs to } S\}\), \(f_r(x^1) > 0\) and the price vector \(p^1\) is such that \(p^1 \cdot x^1 > 0\); then

\[
f_r(x^1)/f_r(x^0) = Q_r(p_0^0, p_i^0; x_0^0, x^1).
\]

Thus the quadratic mean of order \(r\) quantity index \(Q_r\) is exact for the quadratic mean of order \(r\) aggregator function, which in view of Theorem (4.3) implies that \(Q_r\) is a superlative index number.

Suppose that \(x^s \gg 0_N\) is a solution to \(\max_x \{f_r(x) : p^s \cdot x \leq p^s \cdot x^s, x \text{ belongs to } S\}\), where \(f_r(x) > 0, p^s \cdot x^s > 0\) for \(s = 0, 1, 2\). Then using (4.9) three times, we find that

\[
Q_r(p_0^0, p_i^0; x_0^0, x^1)Q_r(p_0^1, p_i^1; x_1^0, x^2) = f_r(x^1)[f_r(x^0)]^{-1} f_r(x^2)[f_r(x^1)]^{-1}
\]

\[
= f_r(x^2)/f_r(x^0)
\]

\[
= Q_r(p_0^0, p_i^0; x_0^0, x^2).
\]

Thus under the assumption that the producer or consumer is maximizing \(f_r(x)\) subject to an expenditure constraint each period, we find that \(Q_r\) will satisfy the circularity test in addition to the other Fisher tests which it satisfies.

A similar proposition is true for any exact index number, a fact which was first noted by Samuelson and Swamy [1974]. Since the circularity test is capable of empirical refutation, we see that we can empirically refute the hypothesis that an economic agent is maximizing \(f_r(x)\) subject to an expenditure constraint.

Thus violations of the circularity test could mean either that the economic agent was not engaging in maximizing behavior or that his aggregator function was not \(f_r(x)\).

We note that Theorem (4.8) did not require that all prices be nonnegative; only that quantities be positive. \(f_r\) can also be a transformation function (recall Section 2) which is nondecreasing in inputs and nonincreasing in outputs. Theorem (4.8) will still hold except that prices of other outputs must be indexed negatively while prices of inputs are taken to be positive. The quantity index \(Q_r\) may be used in the context of productivity measurement just as we used the index \(Q_0\) in Section 3. We will return to this topic in Section 5.
Theorem (4.8) tells us that \( f_r \) defined by (4.1) is exact for \( Q_r \) defined by (4.4). However, could there exist a linearly homogeneous functional form \( f \) different from \( f_r \) which is also exact for \( Q_r \)? The answer is no, as the following theorem shows:

**Theorem 4.10.** (Generalization of Byushgens [1925], Koniüs and Byushgens [1926]): Let \( S \) be an open subset of the positive orthant in \( \mathbb{R}^N \) which is also a convex cone. Suppose \( f \) is defined over \( S \) and is (i) positive, (ii) once-differentiable, (iii) linearly homogeneous, and (iv) concave. Suppose that \( f \) is exact for the quantity index \( Q_r \) defined by (4.4) for \( r \neq 0 \) (i.e., if \( x^s \) is a solution to \( \max_x \{ f(x) : p^s \cdot x \leq p^s \cdot x^s, x \text{ belongs to } S \} \) for \( s = 0, 1 \), then \( Q_r(p^0, p^1; x^0, x^1) = f(x^1)/f(x^0) \)). Then \( f \) is a quadratic mean of order \( r \) defined by (4.1) for some \( a_{ij}, 1 \leq i \leq j \leq N \).

We note that the functional form \( f_r \) defined by (4.1) may also be used as a factor requirements function, and that the quantity index \( Q_r \) defined by (4.4) will still be exact for \( f_r \); i.e., Theorems (4.8) and (4.10) will still hold except that the maximization problems \( \max_x \{ f_r(x) : p^s \cdot x \leq p^s \cdot x^s, x \text{ belongs to } S \} \) are replaced by the minimization problems \( \min_x \{ f_r(x) : p^s \cdot x \geq p^s \cdot x^s, x \text{ belongs to } S \} \) for \( s = 0, 1 \), and condition (iv) is changed from concavity to convexity. Thus the quadratic mean of order \( r \) indexes \( Q_r \) can be used to aggregate either inputs or outputs provided that the functional form for the aggregator function is a quadratic mean of order \( r \).

The above theorems have their counterparts in the dual space.

**Theorem 4.11.** Suppose that (i) \( c_r(p) \equiv (\sum_i \sum_j b_{ij} p_i^{r/2} p_j^{1/2})^{1/r} \), where \( b_{ij} = b_{ji} \) for all \( i, j, r \neq 0 \) and \( (p_1, p_2, \ldots, p_N) \equiv p \) belongs to \( S \) where \( S \) is an open, convex cone which is a subset of the positive orthant in \( \mathbb{R}^N \); (ii) \( c_r(p) \) is positive, linearly homogeneous and concave over \( S \); (iii) \( x^0 \cdot p^0 \cdot x^0 = \nabla c_r(p^0)/c_r(p^0) \), where \( p^0 \gg 0 \) so that (using the corollary (2.14) to Shephard’s lemma) \( x^0 \) is a solution to the aggregator maximization problem \( \max_x \{ f_r(x) : p^0 \cdot x \leq p^0 \cdot x^0, x \geq 0 \} \), where \( f_r \) is the direct aggregator function which is dual to \( c_r(p) \); and (iv) \( x^1 \cdot p^1 \cdot x^1 = \nabla c_r(p^1)/c_r(p^1), p^1 \gg 0 \) so that \( x^1 \) is a solution to the aggregator maximization problem \( \max_x \{ f_r(x) : p^1 \cdot x \leq p^1 \cdot x^1, x \geq 0 \} \).

Then

\[
(4.12) \quad c_r(p^1)/c_r(p^0) = P_r(p^0, p^1, x^0, x^1),
\]

where \( P_r \) is the quadratic mean of order \( r \) price index defined by (4.5).

The proof of Theorem (4.11) is analogous to the proof of Theorem (4.8), except that \( p \) replaces \( x \), \( c_r \) replaces \( f_r \), and Corollary (2.14) is used instead of the Koniüs–Byushgens–Hotelling Lemma.

Thus the quadratic mean of order \( r \) unit cost function \( c_r \) is exact for the price index \( P_r \). Since, by Theorem (4.3), \( c_r \) can provide a second order approximation to an arbitrary twice differentiable unit cost function, we see that \( P_r \) is a superlative price index for each \( r \neq 0 \). We note also that there is an analogue to Theorem (4.10) for \( P_r \); i.e., \( c_r \) is essentially the only functional form which is exact for the price index function \( P_r \).

However, if we relax the assumption that the underlying aggregator function be linearly homogeneous, then the index numbers \( P_r \) and \( Q_r \) can be exact for a number of true cost of living price indexes and Malmquist quantity indexes, respectively; i.e., analogues to Theorems (2.16) and (2.17) hold.

We have obtained two families of price and quantity indexes: \( P_r, \tilde{Q}_r \), and \( \tilde{P}_r, Q_r \), defined by (4.6) and (4.4) for \( r \neq 0 \). The first price-quantity family corresponds to an aggregator function \( f_r \) which has the unit cost function \( c_r \) defined by (4.2) as its dual, and the second price-quantity family corresponds to an aggregator function \( f_r \) defined by (4.1). Recall also that the price-quantity indexes \( P_0, Q_0 \) correspond to a translog unit cost function, while \( P_0, Q_0 \) correspond to a homogeneous translog aggregator function.

For various values of \( r \), some of the indexes \( P_r \) or \( \tilde{P}_r \) have been considered in the literature. For \( r = 2, P_2 \equiv P_2 \) becomes the Pigou [1912] and Fisher [1922] ideal price index which corresponds to the Koniüs–Byushgens [1926] homogeneous quadratic aggregator function \( f(x) \equiv (x^T Ax)^{1/2} \), where \( A = A^T \) is a symmetric matrix of coefficients and it also corresponds to the unit cost function \( c_2(p) \equiv (p^T B p)^{1/2} \), where \( B = B^T \) is a symmetric matrix of coefficients. If \( A^{-1} \) exists, then it is easy to show that the unit cost function which is dual to \( f_2 \) is \( \tilde{c}_2(p) = (p^T A^{-1} p)^{1/2} \) (at least for a range of prices). However, if \( f_2(x) \equiv (x^T a a^T x)^{1/2} = a^T x \), where \( a \gg 0 \) is a vector of coefficients (linear aggregator function), then \( \tilde{c}_2(p) = \tilde{c}_2(p_1, p_2, \ldots, p_N) \equiv \min_i \{ p_i / a_i : i = 1, 2, \ldots, N \} \) which is not a member of the family of unit cost functions defined by \( c_2(p) \equiv (p^T B p)^{1/2} \). On the other hand, if \( c_2(p) = (p^T b b^T p)^{1/2} = b^T p \), where \( b \gg 0 \) is a vector of coefficients (Leontief unit cost function), then the dual aggregator function is \( f_2(x_1, x_2, \ldots, x_N) = \min_i \{ x_i / b_i : i = 1, 2, \ldots, N \} \), which is a Leontief aggregator function. Thus \( P_2 \) is exact for a Leontief aggregator function (a fact which was noted by Pollak [1971a]) and, since \( P_2 \equiv P_2 \), it is also exact for a linear aggregator function. This is an extremely useful property for an index number formula, since the two types of aggregator functions correspond to zero substitutability between the commodities to be aggregated and infinite substitutability, respectively.

For \( r = 1 \), the price index \( P_1 \) has been recommended by Walsh [1901; 105]. \( P_1 \) is exact for the unit cost function \( c_1 \), whose dual \( f_1 \) is the generalized Leontief aggregator function (see Diewert [1971a] for the definition of this function) which has the Leontief aggregator function as a special case. Walsh also recommended the price index \( P_1 \), which is exact for the generalized linear
aggregator function $f_1$ (see Diewert [1971a] for the definition of this function) which of course has the linear aggregator function as a special case.

If, in fact, a producer or consumer were maximizing a linear homogeneous function subject to an expenditure constraint for a number of time periods, we would expect (in view of the Approximation Theorem (4.3)) that the price indexes $P_r$ and $\tilde{P}_r$ should more or less coincide, particularly if the variation in relative prices were small. However, since real world data are not necessarily consistent with this maximization hypothesis, let us consider some empirical evidence on this point.

Table 1. Comparison of Some Index Numbers Tabulated by Fisher

<table>
<thead>
<tr>
<th>Price index</th>
<th>Fisher number</th>
<th>1913</th>
<th>1914</th>
<th>1915</th>
<th>1916</th>
<th>1917</th>
<th>1918</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{Pa}$</td>
<td>54</td>
<td>100</td>
<td>100.3</td>
<td>100.1</td>
<td>114.4</td>
<td>161.1</td>
<td>177.4</td>
</tr>
<tr>
<td>$P_{La}$</td>
<td>53</td>
<td>100</td>
<td>99.9</td>
<td>99.7</td>
<td>114.1</td>
<td>162.1</td>
<td>177.9</td>
</tr>
<tr>
<td>$P_0$</td>
<td>123</td>
<td>100</td>
<td>100.1</td>
<td>99.9</td>
<td>113.8</td>
<td>162.1</td>
<td>177.8</td>
</tr>
<tr>
<td>$P_1$</td>
<td>124</td>
<td>100</td>
<td>100.16</td>
<td>99.85</td>
<td>114.25</td>
<td>161.74</td>
<td>178.16</td>
</tr>
<tr>
<td>$P_2$</td>
<td>1153</td>
<td>100</td>
<td>100.13</td>
<td>99.89</td>
<td>114.20</td>
<td>161.70</td>
<td>177.83</td>
</tr>
<tr>
<td>$P_3$</td>
<td>1154</td>
<td>100</td>
<td>100.12</td>
<td>99.90</td>
<td>114.24</td>
<td>161.73</td>
<td>177.76</td>
</tr>
<tr>
<td>$P_4$</td>
<td>353</td>
<td>100</td>
<td>100.12</td>
<td>99.89</td>
<td>114.21</td>
<td>161.56</td>
<td>177.65</td>
</tr>
</tbody>
</table>

Irving Fisher [1922; 489] tabled the wholesale prices and the quantities marketed for 36 primary commodities in the U.S. during the war years (1913–1918), a time of very rapid price and quantity changes. Fisher calculated and compared 134 different price indexes using this data. Table 1 reproduces Fisher’s [1922; 244–247] computations for the Paasche and Laspeyres price indexes, $P_{Pa}$ and $P_{La}$, as well as for $P_0$, $P_1$, $P_2$, and $P_2 = P_2 = P_{14}$. Fisher’s identification number is given in column 2 of the table; e.g. $P_2$ or the ‘ideal’ price index was identified as number 353 by Fisher. All index numbers were calculated using 1913 as a base.

Note that the Paasche and Laspeyres indexes coincide to about two significant figures, while the last four indexes mostly lie between the Paasche and Laspeyres indexes and coincide to three significant figures. Fisher [1922; 278] also calculated $P_2$ (and some of the other ‘very good’ index numbers) using different years as the base year and then he compared how the various series differed; that is, he tested for ‘circularity’. Fisher found that the average discrepancy was only about 1/3 percent between any two bases. Thus as far as Fisher’s time series data are concerned, it appears that any one of the price indexes, $P_r$ or $\tilde{P}_r$, gives the same answer to three significant figures, and that violations of circularity are only about 1/3 percent so that the choice of base year is not too important.$^{16}$

To determine how the price indexes $P_r$ compare for different $r$’s in the context of cross section data, one may look at Ruggles’ [1967; 189–190] paper which compares the consumer price indexes $P_{Pa}$, $P_{La}$, $P_0$ and $P_2$ for 19 Latin American countries for the year 1961. The indexes $P_0$ and $P_2$, using Argentina as a base, differed by about one percent per observation, while $P_0$ and $P_2$, using Venezuela as a base (the relative prices in the two countries differed markedly), differed by about 1.5 percent per observation. $P_2$ failed the circularity test (comparing values with Venezuela and Argentina as the base country) by an average of about two percent per observation, while $P_0$ failed the circularity test by about three percent per observation.$^{17}$ Thus it appears that the indexes $P_r$ differ more and violate circularity more in the context of cross section analysis than in time series analysis. However, the agreement between $P_0$ and $P_2$ in the cross section context is still remarkable since the Paasche and Laspeyres indexes differed by about 50 percent per observation.

5. Concluding Remarks

We have obtained two families of superlative price and quantity indexes, $(P_r, \tilde{Q}_r)$ and $(\tilde{P}_r, Q_r)$; that is, each of these index numbers is exact for a homogeneous aggregator function which is capable of providing a second order approximation to an arbitrary twice continuously differentiable aggregator function or its dual unit cost function. Moreover, $(P_r, \tilde{Q}_r)$ and $(\tilde{P}_r, Q_r)$ satisfy many of the Irving Fisher tests for index numbers in addition to their being consistent with a homogeneous aggregator function. Note also that if prices are varying proportionately, then the aggregates $Q_r$ and $Q_r$ are consistent with Hicks’ [1946] aggregation theorem.

$^{16}$However, as a matter of general principle, it would seem that the chain method of calculating index numbers would be preferable, since over longer periods of time, the underlying functional form for the aggregator function may gradually change, so that, for example, (1.5) will only be approximately satisfied, with the degree of approximation becoming better as $r$ approaches 0.

$^{17}$This failure of the circularity test should not be too surprising from the viewpoint of economic theory since we do not expect the aggregator function for the 270 consumer goods and services to be representable as a linearly homogeneous function; that is, we do not expect all ‘income’ or expenditure elasticities to be unitary.
Although any one of the index number pairs, \((P_r, Q_r)\) or \((\bar{P}_r, \bar{Q}_r)\) could be used in empirical applications, we would recommend the use of
\[
(P_2, Q_2) \equiv (\bar{P}_2, \bar{Q}_2) \equiv \left\{ \left( p^1 x^1 p^1 x^0/p^1 x^0, \lambda^0 (x^0) \right), \left( x^1 p^1 x^0/p^0 x^0, \lambda^0 (x^0) \right) \right\}^{1/2},
\]
Irving Fisher’s [1922] ideal index numbers, as the preferred pair of index numbers. There are at least three reasons for this selection.

(i) The functional form for the Fisher-Konius-Byuhsens ideal index number is particularly simple and this leads to certain simplifications in applications. For example, recall equation (3.18) which we used in order to measure technical progress, \((1 + \tau)\), in an economy whose transformation function could be represented by a (nonseparable) homogeneous translog transformation function \(t\). If we assume \(t(z) = f_2(z)\), where \(f_2\) is defined by (4.1) for \(r = 2\), then the analogue to (3.18) is
\[
\frac{y_1}{y_1^0} = \frac{\left[ - \bar{w}^1 \cdot \bar{y}^1 + p^1 \cdot (1 + \tau) x^1 \right]}{(\bar{w}^1 \cdot \bar{y}^0 + p^1 \cdot x^0)(\bar{w}^0 \cdot \bar{y}^0 + p^0 \cdot x^0)}.
\]

If we square both sides of (5.1), the resulting quadratic equation in \((1 + \tau)\) can easily be solved, given market data.

(ii) The indexes \(P_2(p^0, p^1; x^0, x^1)\) and \(Q_2(p^0, p^1; x^0, x^1)\) are functions of \(p_0, x_1/p_0, x_0, x_1\) and \(p_1, x_0/p_1, x_1\), which are ‘sufficient statistics’ for revealed preference theory, and moreover \(Q_2\) is consistent with revealed preference theory in the following sense: (a) if \(p^0 \cdot x^1 < p^0 \cdot x_0\) and \(p^1 \cdot x_0 \geq p^1 \cdot x^1\) (i.e., \(x_0\) revealed preferred to \(x^1\)), then \(Q_2(p^0, p^1; x^0, x^1) < 1\); (b) if \(p^0 \cdot x^1 \geq p^0 \cdot x_0\) and \(p^1 \cdot x_0 < p^1 \cdot x^1\) (i.e., \(x^1\) revealed preferred to \(x_0\)), then \(Q_2(p^0, p^1; x^0, x^1) > 1\) (i.e., the quantity index indicates an increase in the aggregate); and (c) if \(p^0 \cdot x^1 = p^0 \cdot x_0\) and \(p^1 \cdot x_0 = p^1 \cdot x^1\) (i.e., \(x^0\) and \(x_0\) revealed to be equivalent or indifferent), then \(Q_2(p^0, p^1; x^0, x^1) = 1\) (i.e., the quantity index remains unchanged). Thus even if the true aggregator function \(f\) is nonhomothetic, the quantity index \(Q_2\) will correctly indicate the direction of change in the aggregate when revealed preference theory tells us that the aggregate is decreasing, increasing or remaining constant.

(iii) The index number pair \((P_2, Q_2)\) is consistent with both a linear aggregator function (infinite substitutability between the goods to be aggregated) and a Leontief aggregator function (zero substitutability between the commodities to be aggregated). No other \((P_r, Q_r)\) or \((\bar{P}_r, \bar{Q}_r)\) has this very useful property.

6. Proofs of Theorems

**Proof of (2.2).**

\[
f(z^1) - f(z^0) = a^T z^1 + \frac{1}{2} z^1 A z^1 - a^T z^0 - \frac{1}{2} z^0 A z^0
\]
\[
= a^T (z^1 - z^0) + \frac{1}{2} z^1 A(z^1 - z^0) + \frac{1}{2} z^0 A(z^1 - z^0)
\]
\[
= \frac{1}{2} (a + A z^1 + a + A z^0) (z^1 - z^0), \quad \text{since } A = A^T
\]
\[
= \frac{1}{2} (\nabla f(z^1) + \nabla f(z^0))(z^1 - z^0).
\]

Assume \(f\) is thrice differentiable and satisfies the functional equation \(f(x) - f(y) = \frac{1}{2} \nabla f(x) + \nabla f(y))''(x - y)\), for all \(x\) and \(y\), in an open neighborhood. We wish to find the function that is characterized by the fact that its average slope between any two points equals the average of the endpoint slopes in the direction defined by the difference between the two points. If \(f\) is a function of one variable, the functional equation becomes \(f(x) - f(y) = \frac{1}{2} f''(x) + f''(y))(x - y)\). If we differentiate this last equation twice with respect to \(x\), we obtain the differential equation \(\frac{1}{2} f'''(x)(x - y) = 0\), which implies that \(f(x)\) is a polynomial of degree two. The general case follows in an analogous manner using the directional derivative concept.

**Proof of (2.4).** \(\lambda_1^*\) and \(x_1^*\) will satisfy the first order necessary conditions for an interior maximum for the maximization problem (2.5),

\[
\nabla f(x^1) = \lambda_1^* p^1; \quad p_1 \cdot x_1 = Y^1.
\]

Similarly, \(\lambda_0^*\) and \(x_0^*\) will satisfy the first order conditions for the constrained maximization problem (2.6),

\[
\nabla f(x^0) = \lambda_0^* p^0; \quad p_0 \cdot x_0 = Y^0.
\]

Now substitute the first part of (6.1) into the right hand side of the identity (2.3), and obtain (2.7).

**Proof of (2.16).** For a fixed \(u^*\), \(\ln C(u^*; p)\) is quadratic in the vector of variables \(\ln p\) and we may apply the quadratic approximation Lemma (2.2) to
obtain
\[
\ln C(u^*, p^1) - \ln C(u^*, p^0) \\
= \frac{1}{2} \left[ \overline{p} \left( \nabla C(u^*, p^1) - \nabla C(u^*, p^0) \right) \right] \cdot \ln (p^1 - p^0) \\
= \frac{1}{2} \left[ \overline{p} \left( \nabla C(u^*, p^1) - \nabla C(u^*, p^0) \right) \right] \cdot \ln (p^1 - p^0)
\]
where the equality follows upon evaluating the derivatives of \( C \) and noting that
\[
2 \ln u^* = \ln u^1 + \ln u^0,
\]
\[
= \ln P_0(p^0, p^1, x^0, x^1)
\]
using the definitions of \( x^0 = \nabla C(u^0, p^0) \), \( x^1 = \nabla C(u^1, p^1) \) and \( P_0 \).

**Proof of (2.17).** It is first necessary to express the partial derivatives of \( D \) with respect to the components of \( x, \nabla x D(u^r, x^r), r = 0, 1 \), in terms of the partial derivatives of \( f \). We have \( D(u^r, x^r) \equiv \max_k \{ k : f(x^r/k) \geq u^r \} = 1 \), for \( r = 0, 1 \), since each \( x^r \) is on the \( u^r \)-utility surface. To find out how the distance \( D(u^0, x^0) \) changes as the components of \( x^0 \) change, apply the implicit function theorem to the equation \( f(x^0/k) = u^0 \) (where \( k = 1 \) initially). We find that
\[
\frac{\partial k}{\partial x_j} \equiv \frac{\partial D(u^0, x^0)}{\partial x_j} = f_j(x^0) / \sum_{k=1}^{N} x_k^0 f_k(x^0), \quad j = 1, 2, \ldots, N.
\]
Similarly
\[
\frac{\partial D(u^1, x^1)}{\partial x_j} = f_j(x^1) / \sum_{k=1}^{N} x_k^1 f_k(x^1), \quad j = 1, 2, \ldots, N.
\]
Furthermore, the first order conditions for the two aggregator maximization problems after elimination of the Lagrange multipliers yield the relations
\[
p_j^0/p^0 \cdot x^0 = f_j(x^0) / \sum_{k=1}^{N} x_k^0 f_k(x^0), \quad j = 1, 2, \ldots, N,
\]
and
\[
p_j^1/p^1 \cdot x^1 = f_j(x^1) / \sum_{k=1}^{N} x_k^1 f_k(x^1), \quad j = 1, 2, \ldots, N.
\]

Upon noting that the right hand sides of the last set of relations are identical to the right hand sides of the earlier relations, we obtain
\[
\frac{\nabla x D(u^0, x^0)}{D(u^0, x^0)} = \frac{p^0}{p^1} \cdot x^0 \quad \text{and} \quad \frac{\nabla x D(u^1, x^1)}{D(u^1, x^1)} = \frac{p^1}{p^0} \cdot x^1.
\]

Now for a fixed \( u^* \), \( \ln D(u^*, x) \) is quadratic in the vector of variables \( \ln x \) and we may again apply the Quadratic Approximation Lemma (2.2) to obtain the following equality:
\[
\ln D(u^*, x^1) - \ln D(u^*, x^0) \\
= \frac{1}{2} \left[ \overline{x} \overline{\nabla_x D(u^*, x^1)} + x^0 \overline{\nabla_x D(u^*, x^0)} \right] \cdot (\ln x^1 - \ln x^0) \\
= \ln Q_0(p^0, p^1, x^0, x^1),
\]
where the equality follows upon evaluating the derivatives of \( D \), noting that
\[
2 \ln u^* = \ln u^1 + \ln u^0,
\]
using (6.2), the equalities \( D(u^1, x^1) = 1 \), \( D(u^0, x^0) = 1 \) and the definition of \( Q_0 \).

**Proof of (4.3).** Since both \( f \) and \( f_r \) are twice continuously differentiable, their Hessian matrices evaluated at \( x^r, \nabla^2 f(x^r) \) and \( \nabla^2 f_r(x^r) \), are both symmetric. Thus we need only show that \( \partial^2 f(x^r)/\partial x_i \partial x_j = \partial^2 f_r(x^r)/\partial x_i \partial x_j, \) for \( 1 \leq i \leq j \leq N \). Furthermore, by Euler’s theorem on linearly homogeneous functions, \( f(x^r) = x^i \nabla f(x^r) \) and \( f_r(x^r) = x^i \nabla f_r(x^r) \). Since the partial derivative functions \( \partial f(x)/\partial x_i \) are homogeneous of degree zero, application of Euler’s theorem on homogeneous functions yields, for \( i = 1, 2, \ldots, N \),
\[
\sum_{j=1}^{N} x_j^r \partial^2 f(x^r)/\partial x_i \partial x_j = 0 = \sum_{j=1}^{N} x_j^r \partial^2 f_r(x^r)/\partial x_i \partial x_j.
\]
Thus the above material implies that \( f_r(x^r) = f(x^r), \) \( \nabla f_r(x^r) = \nabla f(x^r) \) and \( \nabla^2 f_r(x^r) = \nabla^2 f(x^r) \) will be satisfied under our present hypotheses if and only if
\[
\partial f_r(x^r)/\partial x_i = f_r^i \equiv \partial f(x^r)/\partial x_i, \quad \text{for} \ 1 \leq i \leq j \leq N.
\]

Thus we need to choose the \( N(N+1)/2 \) independent parameters \( a_{ij}(1 \leq i \leq j \leq N) \), so that the \( N + N(N - 1)/2 = N(N + 1)/2 \) equations (6.4) and (6.5) are satisfied. Recall that \( x^* \equiv (x_1^*, x_2^*, \ldots, x_N^*) \gg 0_N \) and that
\[
y^* \equiv x^* \nabla f(x^*) = f(x^*) > 0, \text{ since } f \text{ is assumed to be positive over its}
domain of definition. Thus since \( y^* > 0 \), \( x_i^* > 0 \) and \( r \neq 0 \), the numbers \( a_{ij}^* \), for \( 1 \leq i < j \leq N \), can be defined by solving the following equations for \( a_{ij}^* \):

\[
(6.6) \quad f_{ij}^* = \frac{1-r}{y^*} f_{i}^* f_{j}^* + \frac{r}{2} y^* (1-r) a_{ij}^* x_i^{r/2-1} x_j^{r/2-1}, \quad 1 \leq i < j \leq N.
\]

The system of equation (6.6) is equivalent to (6.5) if we also make use of (6.4). Now define \( a_{ij}^* = a_{ji}^* \), for \( i \neq j \), and then \( a_{ii}^* \) is defined as the solution to the following equation:

\[
(6.7) \quad \sum_{j=1}^{N} a_{ij}^* x_i^{r/2-1} x_j^{r/2} y^*(1-r) = f_i^*, \quad i = 1, 2, \ldots, N.
\]

Now define \( f_r(x) \equiv \left( \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^* x_i^{r/2} x_j^{r/2} \right)^{1/r} \), and it can be verified readily that equations (6.4) and (6.5) are satisfied by \( f_r \) as defined.

Proof of (4.8). Using assumptions (ii) and (iii) of (4.8) yields

\[
(6.8) \quad v^0 \equiv p^0/p^0 \cdot x^0 = \nabla f_r(x^0)/f_r(x^0),
\]

\[
(6.9) \quad v^1 \equiv p^1/p^1 \cdot x^1 = \nabla f_r(x^1)/f_r(x^1).
\]

Upon differentiating \( f_r(x^0) \), the \( i \)th equation in (6.8) becomes

\[
v_i^0 \equiv p_i^0/p^0 \cdot x_i^0 = (x_i^1)^{(r/2)-1} \sum_{j=1}^{N} a_{ij} x_j^{r/2} / \sum_{k=1}^{N} a_{km} x_k^{r/2} x_m^{r/2},
\]

and therefore

\[
(6.10) \quad \sum_{i=1}^{N} x_i^{1(r/2)} v_i^0 x_i^{1-(r/2)} = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} x_j^{r/2} / \sum_{k=1}^{N} a_{km} x_k^{r/2} x_m^{r/2}.
\]

Similarly, using equation (6.9), we obtain

\[
(6.11) \quad \sum_{i=1}^{N} x_i^{0(r/2)} v_i^1 x_i^{1-(r/2)} = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} x_j^{r/2} / \sum_{k=1}^{N} a_{km} x_k^{r/2} x_m^{r/2}.
\]

Upon noting that \( a_{ij} = a_{ji} \), take the ratio of (6.10) to (6.11),

\[
(6.12) \quad \frac{\sum_{i=1}^{N} (x_i^1/x_i^0)^{r/2} a_{ij} x_j^{r/2}}{\sum_{j=1}^{N} (x_j^0/x_j^1)^{r/2} a_{ij} x_j^{r/2}} = \left[ \frac{f_r(x^1)}{f_r(x^0)} \right]^r.
\]

Take the \( r \)th root of both sides of (6.12) and obtain (4.9).

Proof of (4.10). Let \( x, y \) be any two points belonging to \( S \) such that

\[
(6.13) \quad 1 = f(x) = f(y) = x \cdot \nabla f(x) = y \cdot \nabla f(y),
\]

where the last two equalities follow from the linear homogeneity of \( f \). Since \( f \) is a concave function over \( S \), for every \( z \) belonging to \( S \), \( f(z) \leq f(x) + \nabla f(x) \cdot (z-x) = f(x) + \nabla f(x) \cdot z - f(x) = \nabla f(x) \cdot z \), and similarly \( f(z) \leq \nabla f(y) \cdot z \). Thus \( x \) is a solution to \( \max \{ f(z) : \nabla f(x) \cdot z \leq \nabla f(y) \cdot z \} \), and \( y \) is a solution to \( \max \{ f(z) : \nabla f(y) \cdot z \leq \nabla f(y) \cdot y \} \). Since \( f \) is exact for \( Q_r \) for some \( r \neq 0 \) by assumption, we must have, using (6.13),

\[
Q_r(\nabla f(x), \nabla f(y); x, y) = f(y)/f(x) = 1,
\]

or

\[
(6.14) \quad \sum_{i=1}^{N} (x_i/y_i)^{r/2} f_i(y) y_i / \nabla f(y) \cdot y = \sum_{i=1}^{N} (y_i/x_i)^{r/2} f_i(x) x_i / \nabla f(x) \cdot x,
\]

and since we can choose the vectors \( y^1, y^2, \ldots, y^N \) to be such that the coefficient matrix on the left hand side of the system of \( N \) equations is nonsingular, we may invert the coefficient matrix and obtain the solution

\[
(6.15) \quad f_n(x) x_1^{1-(r/2)} = \sum_{j=1}^{N} A_{nj} x_j^{r/2}, \quad n = 1, 2, \ldots, N,
\]

for some constants, \( A_{ij} \), \( 1 \leq i, j \leq N \). Equation (6.15) is valid for any \( x \) belonging to \( S \), such that \( f(x) = 1 \); in particular, (6.15) is true for \( x = y \),

\[
(6.16) \quad f_n(y) y_1^{1-(r/2)} = \sum_{j=1}^{N} A_{nj} y_j^{r/2}, \quad n = 1, 2, \ldots, N.
\]
Now substituting (6.15) into the left hand side of (6.14) and (6.16) into the right hand side of (6.14), we obtain

\[ \sum_{n=1}^{N} y_n^{r/2} \sum_{j=1}^{N} A_{nj} x_j^{r/2} = \sum_{n=1}^{N} x_n^{r/2} \sum_{j=1}^{N} A_{nj} y_n^{r/2}, \]

or

\[ \sum_{n} \sum_{j} y_n^{r/2} A_{nj} x_j^{r/2} = \sum_{n} \sum_{j} x_n^{r/2} A_{nj} y_j^{r/2}. \]

Since (6.17) is true for every \( x, y \), such that \( f(x) = 1 = f(y) \), we must have

\[ A_{nj} = A_{jn}, \quad \text{for} \quad 1 \leq n, j \leq N. \]

Now take \( x_n^{r/2} \) times (6.15) and sum over \( n \),

\[ \sum_{n=1}^{N} x_n^{r/2} f_n(x) x_n^{1-(r/2)} = \sum_{n=1}^{N} \sum_{j=1}^{N} A_{nj} x_n^{r/2} x_j^{r/2} = 1, \]

since \( x \cdot \nabla f(x) = f(x) = 1 \).

Thus if \( f(x) = 1 \), then \( x \) satisfies the equation \( \sum_{n} \sum_{j} A_{nj} x_n^{r/2} x_j^{r/2} = 1 \), where \( A_{nj} = A_{jn} \). Since \( f \) is linearly homogeneous by assumption, we must have for \( x \) belonging to \( S \),

\[ f(x) = \left[ \sum_{n=1}^{N} \sum_{j=1}^{N} A_{nj} x_n^{r/2} x_j^{r/2} \right]^{1/r}. \]

References for Chapter 8


Essays in Index Number Theory

8. Exact and Superlative Index Numbers


