Directed Search

Lecture 5: Monetary Economics

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Main sources of this lecture:


1. Transactions Cost and Monetary Policy

\[ m/p = \left[y b/(2 i)\right]^{0.5} \]

\[ 2m = py/N \]

\[ \min[bN + i m/p] \]

Baumol-Tobin inventory model of money
An important feature: transactions cost $\Rightarrow$ limited participation in financial markets

Policy implication: limited participation $\Rightarrow$ open market operations can affect real activity

Examples:

Grossman and Weiss (83), Rotemberg (84)
Lucas (90), Alvarez, et al. (02)
But the real effect can be short-lived!

\[ \frac{m'}{p'} = \frac{m}{p} \]
\[ t' = t \]
\[ N' = N \]

\[ t + (N' - 1)t' = 1 \]

(in contrast to VAR evidence by Christiano et al., 99)
Staggered participation and persistent effect
Our interpretation:

A non-degenerate distribution of money holdings is critical for money injection to have persistent real effects

The objective:
to give a tractable characterization of a monetary eqm in which

• there is microfoundation of money, and

• money distribution is non-degenerate
• Search theory is a natural framework for both:
  – microfoundation of money: decentralized exchange with lack of double coincidence of wants; anonymity

  – exchange generates distribution of money holdings:

\[
m \mapsto \begin{cases}
  \text{no match: } m \\
  \text{match} \\
  \text{match} \\
  \text{buyer’s money: } m - x \\
  \text{seller’s money: } x
\end{cases}
\]
...but challenging to characterize such an equilibrium: distribution is endogenous state with a large dimension

individuals’ decisions, trading prob. \(\overset{\text{state variable}}{\leftrightarrow}\) aggregation \(\overset{\text{distribution of individuals}}{\rightarrow}\) over money

Even numerical computation can be challenging: Molico (06), Chiu and Molico (08)
Generations of money search models:

- 1st generation (Kiyotaki-Wright 89) assumes: all holdings are either zero or one unit

- 2nd generation:
  - Shi (95)-Trejos-Wright (95): indivisible $m$ and divisible goods
  - Green-Zhou (98): discrete $m$ and indivisible goods

- 3rd generation: divisible goods and $m$
  - Shi (97): a larger number of members in each household
  - Lagos-Wright (05): centralized mkt with quasi-linear pref.

We characterize equilibrium without these assumptions.
2. The Model

Model Environment

- large numbers of types of perishable goods, $i \in I$; each is produced by a large number of individuals

- no double coincidence of wants:
  
  type $i$ consumes good $i$ but produces good $i + 1$.

- anonymity: no record keeping

- fiat money: intrinsically useless; stock = $M$
• firms: $I$ types
  
  – production: $q$ of type $(i - 1)$ labor $\implies q$ of type $i$ goods

  – selling: $k$ of type $(i - 1)$ labor $\implies$ one trading post

• competitive labor market: monetary wage rate $= \omega \ M$

• individuals own firms through a diversified mutual fund

• numeraire: labor
Events in a period:

| lotteries on money | chooses to be a worker or a buyer | markets open; search & match; consume |

A worker’s decision: policy function \( \ell^*(m) \in [0, 1] \) solves

\[
W(m) = \max [\beta V(m + \ell) - h(\ell)]
\]

\( m \): real balance; \( V \): ex ante value function
Directed search in the goods market:

Buyers and firms choose which submarket to enter.

A continuum of submarkets \((x, q)\) for each type \(i\) good:

- submarket \((x, q)\): \(x\) real balances for \(q\) units of goods.

- market tightness \(\theta(x, q)\): \(\frac{\# \text{ trading posts}}{\# \text{ buyers}}\)

- matching probability in submarket \((x, q)\):

  a buyer: \(b(x, q) = \lambda(\theta(x, q)), \quad \lambda'(\theta) > 0,\)

  a post: \(s(x, q) = \rho(\theta(x, q)), \quad \rho(\theta) = \lambda(\theta)/\theta\)
A buyer’s decisions:

• chooses which submarket \((x, q)\) to enter

\[
B(m) = \beta V(m) + \max \lambda(\theta(x, q)) [U(q) + \beta V(m - x) - \beta V(m)]
\]

surplus from trade

s.t. \(x \in [0, m], \quad q \geq 0.\)

• policy functions:

  quantity of goods bought: \(q^*(m)\)
  real balance spent: \(x^*(m)\)
  residual balance: \(\phi(m) \equiv m - x^*(m)\)
  trading probability: \(b^*(m) = \lambda(\theta(x^*(m), q^*(m)))\)
A firm’s decisions:

- demand for labor to produce and sell goods

- number of trading posts to be created in submarket \((x, q)\):
  \[
  \begin{align*}
  &= \infty, \quad \text{if } \rho(\theta(x, q))(x - q) > k \\
  &= 0, \quad \text{if } \rho(\theta(x, q))(x - q) < k \\
  &\in [0, \infty), \quad \text{if } \rho(\theta(x, q))(x - q) = k
  \end{align*}
  \]

\(\rho(\theta(x, q))\): matching prob for a post in submarket \((x, q)\)
Market tightness function $\theta(x, q)$:

$$\left[ \rho(\theta(x, q))(x - q) \leq k \right. \text{ and } \left. \theta(x, q) \geq 0 \right]$$

with complementary slackness for ALL $(x, q) \in \mathbb{R}_+^2$.

Restrictions on beliefs out of the equilibrium:

some submarkets $(x, q)$ are inactive in equilibrium,
but we still require $\theta(x, q)$ to satisfy the condition above.
Choice of being a buyer or a worker:

at the beginning of every period, an individual chooses:

$$\tilde{V}(m) = \max\{W(m), B(m)\}$$

Lottery \((z_1, z_2, \pi_1, \pi_2)\):

$$V(m) = \max_{(z_1, z_2, \pi_1, \pi_2)} \left[ \pi_1 \tilde{V}(z_1) + \pi_2 \tilde{V}(z_2) \right]$$

s.t. \( \pi_1 z_1 + \pi_2 z_2 = m, \quad \pi_1 + \pi_2 = 1, \quad z_2 \geq z_1, \)

\( \pi_j \in [0, 1] \) and \( z_j \geq 0 \) for \( j = 1, 2 \).

policy functions: \((z_j^*(m), \pi_j^*(m)) \) \( j=1,2 \)
Figure 1. Lotteries and the ex ante value function
Definition of a monetary steady state:

- Block 1:
  - value functions: \((V, W, B)\),
  - policy functions: \((\ell^*, x^*, q^*, z^*, \pi^*)\),
  - and market tightness function \(\theta(x, q)\)

(i) \(W\) and \(\ell^*\) solve a worker’s problem
(ii) \(B\) and \((x^*, q^*)\) solve a buyer’s problem
(iii) \(V\) and \((z^*, \pi^*)\) solve the lottery problem
(iv) \(\theta\) is consistent with free entry of trading posts
• Block 2:
  distribution of real balances: \( G \), and wage rate: \( \omega \)

  (v) \( G \) is ergodic and generated by \((\ell^*, x^*, q^*, z^*, \pi^*, \theta)\)

  (vi) money is valued \((\omega < \infty)\) and

  all money is held: \(1/\omega = \int m \, dG(m)\).
A monetary steady state is block recursive:

block 1: value functions, policy functions, market tightness

block 2: distribution $G$

wage rate $\omega$

block recursivity makes equilibrium tractable:

- state variable in block 1: agent’s own money balance

- block 2 is easy: counting flows
Why is the steady state block recursive?

- directed search  +  free entry of posts.

- No mixing between different $m$: higher $m \Rightarrow$

  higher matching probability $b$ and higher spending $x$;
  higher quantity of goods obtained: $q$

- A buyer with a particular $m$ only cares about
  $(x, q)$ and $\theta$ in the particular submarket he will enter;

- Each submarket is catered to buyers with a particular $m$
  $\Rightarrow \theta$ depends on $(x, q)$ but NOT on the distribution
Eqm is NOT block recursive when search is undirected:

- bargaining on terms of trade:
  - match surplus depends on $m$ in the match
  - distribution of $m$ matters for profit of a post $\implies$
    - tightness $\theta$ depends on distribution $G$;
    - value and policy functions depend on $G$.

- price posting (with undirected search):
  - whether a meeting results in a trade depends on the random buyer’s $m$
  - distribution of $m$ again matters for profit of a post.
3. Equilibrium Value and Policy Functions

A buyer’s problem:

$$B(m) \equiv \max_{(x,b)} \{ F(x, b, m) : x \in [0, m], \ b \in [0, 1] \}$$

$$F(x, b, m) = \beta V(m) + b \times [u(x, b) + \beta V (m - x) - \beta V(m)]$$

- objective function $F$ is not concave in $(x, b, m)$ jointly

- standard approach in dynamic prog. does not work here

But we want to use first-order and envelope conditions
\( \mathcal{C}[0, \bar{m}] : \) continuous, increasing functions on \([0, \bar{m}]\);
\( \mathcal{V}[0, \bar{m}] : \) subset of \( \mathcal{C}[0, \bar{m}] \) with concave functions.

- Assume \( m \leq \bar{m} < \infty \), take any \( V \in \mathcal{V}[0, \bar{m}] \), and prove:
  3.1. worker’s \( W = T_W V \), where \( T_W : \mathcal{V}[0, \bar{m}] \rightarrow \mathcal{V}[0, \bar{m}] \);
  3.2. buyer’s \( B = T_B V \), where \( T_B : \mathcal{V}[0, \bar{m}] \rightarrow \mathcal{C}[0, \bar{m}] \);
    monotone policy function, first-order and envelope conditions.
  3.3. lottery problem \( V = TV : \mathcal{V}[0, \bar{m}] \rightarrow \mathcal{V}[0, \bar{m}] \);
    \( T \) is monotone contraction \( \implies \) unique \( V \in \mathcal{V}[0, \bar{m}] \).

- prove \( m \leq \bar{m} < \infty \), indeed.
Textbook approach (Stokey et al. 98) does not work here:

\[ B(m) = T_B V(m) \equiv \max_{(x,b)} \{ F(x,b,m) : x \in [0,m], \ b \in [0,1] \} \]

\[ F(x,b,m) = \beta V(m) + b \times [u(x,b) + \beta V(m - x) - \beta V(m)] \]

Objective function \( F \) is not concave in \( (x,b,m) \) jointly.

- why cannot we simply assume that \( F \) is concave?
  
  need restrictions on endogenous \( V \) that cannot be verified.

- some other approaches are not applicable either:
  
  e.g., equi-differentiability in \( m \) (Milgrom and Segal 02).
Our approach (related to that in Gonzalez and Shi 10):

\[ B(m) = \max_{(x,b)} \{ F(x, b, m) : x \in [0, m], \ b \in [0, 1] \} \]

(a) adapt **lattice-theoretic approach** (Topkis 98) to prove that policy functions are monotone

(b) use (a) and first principles of calculus to prove \( B \) and \( V \) are differentiable at \( m \) induced by optimal choices

(c) use (b) to validate first-order and envelope conditions

(d) prove that \( V \) is differentiable at all \( m \)
More specifics of step (a): $F(x, b, m)$ is NOT supermodular!

$$F(x, b, m) = \beta V(m) + b \times [u(x, b) + \beta V(m - x) - \beta V(m)]$$

To go around this problem:

- For any given $b$, optimal $x$ solves: $\max_x f(x, b, m)$
  $$f(x, b, m) = u(x, b) + \beta V(m - x).$$

  $f(x, b, m)$ is supermodular $\implies$

  (i) optimal $\tilde{x}(b, m)$ is increasing;
  
  (ii) $\tilde{f}(b, m) = \max_x f(x, b, m)$ is supermodular.

- Optimal $b$ solves: $\max_b \tilde{F}(b, m)$
  $$\tilde{F}(b, m) = (1 - b)\beta V(m) + b \tilde{f}(b, m).$$

  $\tilde{F}(b, m)$ is supermodular $\implies$ optimal $b(m)$ is increasing
4. Monetary Steady State

A buyer’s spending pattern:
Contrast to Baumol-Tobin inventory model of money:

- Transactions cost lies not in getting money, but in spending money: one example is cost $k$ of trading post.

- Endogenous features of trade: A buyer chooses
  - how much money to spend: $x^*(m)$
  - how much consumption to have: $q^*(m)$
  - how quickly to get a trade: $b^*(m)$

- Staggered transaction pattern can be an equilibrium outcome.
Existence and uniqueness of an equilibrium:

• A unique monetary steady state exists and is block recursive

• A lottery may or may not be used in the equilibrium

• If a lottery is used, it is used only at the highest balance, $\hat{m}$:
  – convex disutility of labor $\rightarrow$ need to smooth marginal cost by working for consecutive periods, each with low hours;
  – lottery does the smoothing better
Does the equilibrium produce money dispersion?

- Yes, if individuals are sufficiently patient: $\beta > \beta_0$.

<table>
<thead>
<tr>
<th></th>
<th>spend all $\hat{m}$ once:</th>
<th>in a buying sequence:</th>
</tr>
</thead>
<tbody>
<tr>
<td>pros:</td>
<td>high current $c$,</td>
<td>smoothing in $c$ over periods</td>
</tr>
<tr>
<td>cons:</td>
<td>little smoothing in $c$,</td>
<td>discounting of future $c$</td>
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</tbody>
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- $\beta_0$ is lower (money dispersion is more likely) if
  - utility of consumption is more concave
  - disutility of labor is less convex
  - cost of maintaining trading posts is lower.
Equilibrium distribution of money holdings:

Density of money distribution and trading probability
Effect of a one-time money injection:

• neutral in the steady state

• proportional injection is also neutral in the short run

• other types of injection are NOT neutral in the short run:
  – a lump-sum injection tends to compress distribution and increase average real balance;
  – open market operations are likely to have real effects

• how persistent the real effect is depends on how dispersed money holdings are.
What about studying money growth and inflation?

- equilibrium is still block recursive in the steady state: relatively simple to do comparative statics.

- transitional dynamics are not block recursive, but they depend on distribution only through the scalar $\omega$:
  - given $\omega$, compute decisions and matching prob’s
  - flow equations $\Rightarrow$ next period’s distribution and $\omega_{+1}$
  - iterate on this procedure
We actually did compute the steady state:

**discount factor:** \[ \beta = (1.05)^{-1/4} \]

**utility:** \[ u(q) = \frac{q^{1-\alpha}}{1-\alpha} \quad \alpha = \frac{2}{3} \]

**disutility of labor:** \[ h(l) = \frac{l}{1-l} \]

**cost of post:** \[ k = 0.005 \quad \text{(markup=0.3)} \]

**matching function:** \[ m(N_b, N_s) = \frac{N_b \cdot N_s}{N_b + N_s} \]
Real Balances

Money Growth Rate

Z-average  Z-coeff var  Velocity

39
5. Conclusion

Formalized a monetary theory with non-degenerate distribution:

• directed search induces buyers to sort by $m$
  into markets with different $(b, x, q)$

• block recursive equilibrium with money dispersion

• some tools to do dynamic programming

• potentially persistent real effects of monetary policy

• future work on policy analysis: a lot to do