Limits and topology of metric spaces

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UBC
Economics 526

September 27, 2013
Limits and topology of metric spaces

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Section 1

Sequences and limits
Sequences and limits

- **sequence** is a list of elements, \( \{x_1, x_2, \ldots \} \) or \( \{x_n\}_{n=1}^{\infty} \) or \( \{x_n\} \)
  - Different than set

- Examples
  1. \( \{1, 1, 2, 3, 5, 8, \ldots \} \)
  2. \( \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \} \)
  3. \( \{\frac{1}{2}, \frac{-2}{3}, \frac{3}{4}, \frac{-4}{5}, \frac{5}{6}, \ldots \} \)
Definition
A metric space is a set, $X$, and function $d: X \times X \to \mathbb{R}$ called a metric (or distance) such that $\forall x, y, z \in X$

1. $d(x, y) > 0$ unless $x = y$ and then $d(x, x) = 0$
2. (symmetry) $d(x, y) = d(y, x)$
3. (triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$.

Example
$\mathbb{R}$ is a metric space with $d(x, y) = |x - y|$.

Example
Any normed vector space is a metric space with $d(x, y) = \|x - y\|$.
Definition
A sequence \( \{x_n\}_{n=1}^{\infty} \) in a metric space converges to \( x \) if \( \forall \epsilon > 0 \ \exists N \) such that
\[
d(x_n, x) < \epsilon
\]
for all \( n \geq N \). We call \( x \) the limit of \( \{x_n\}_{n=1}^{\infty} \) and write
\[
\lim_{n \to \infty} x_n = x \text{ or } x_n \to x.
\]
Definition

$a$ is an **accumulation point** of $\{x_n\}_{n=1}^\infty$ if $\forall \epsilon > 0 \ \exists$ infinitely many $x_i$ such that

$$d(a, x_i) < \epsilon.$$
Lemma
If \( x_n \to x \), then \( x \) is an accumulation point of \( \{x_n\}_{n=1}^{\infty} \).

Proof.
Let \( \epsilon > 0 \) be given. By the definition of convergences, \( \exists N \) such that
\[
d(x_n, x) < \epsilon
\]
for all \( n \geq N \). \( \{n \in \mathbb{N} : n \geq N\} \) is infinite, so \( x \) is an accumulation point.
Definition
Given $\{x_n\}_{n=1}^\infty$ and any sequence of positive integers, $\{n_k\}$ such that $n_1 < n_2 < \ldots$ we call $\{x_{n_k}\}$ a subsequence of $\{x_n\}_{n=1}^\infty$.

Lemma
Let $a$ be an accumulation point of $\{x_n\}$. Then $\exists$ a subsequence that converges to $a$. 
Sequences and arithmetic

**Theorem**

Let \( \{x_n\} \) and \( \{y_n\} \) be sequences in a normed vector space \( V \). If \( x_n \to x \) and \( y_n \to y \), then

\[
x_n + y_n \to x + y.
\]

**Proof.**

Let \( \epsilon > 0 \) be given. Then \( \exists N_x \) such that for all \( n \geq N_x \),

\[
d(x_n, x) < \epsilon/2,
\]

and \( \exists N_y \) such that for all \( n \geq N_y \),

\[
d(y_n, y) < \epsilon/2.
\]

Let \( N = \max\{N_x, N_y\} \). Then for all \( n \geq N \),

\[
d(x_n + y_n, x + y) = \|(x_n + y_n) - (x + y)\| \leq \|x_n - x\| + \|y_n - y\| < \epsilon/2 + \epsilon/2 = \epsilon.
\]
Sequences and arithmetic

**Theorem**

Let \( \{x_n\} \) be a sequence in a normed vector space with scalar field \( \mathbb{R} \) and let \( \{c_n\} \) be a sequence in \( \mathbb{R} \). If \( x_n \to x \) and \( c_n \to c \) then

\[
  x_n c_n \to xc.
\]

**Proof.**

On problem set.
**Definition**

Let \( \{x_n\}_{n=1}^{\infty} \) be a sequence in a normed vector space. Let \( s_n = \sum_{i=1}^{n} x_i \) denote the sum of the first \( n \) elements of the sequence. We call \( s_n \) the \( n \)th partial sum. We define the sum of all the \( x_i \)s as

\[
\sum_{i=1}^{\infty} x_i \equiv \lim_{n \to \infty} s_n
\]

This is called a(n infinite) **series**.
Series

Example

Geometric series: \( \sum_{i=0}^{\infty} \beta^i \) where \( \beta \in \mathbb{R} \) has partial sums:

\[
\begin{align*}
S_n &= 1 + \beta + \beta^2 + \cdots + \beta^n \\
&= 1 + \beta(1 + \beta + \cdots + \beta^{n-1}) \\
&= 1 + \beta(1 + \beta + \cdots + \beta^{n-1} + \beta^n) - \beta^{n+1} \\
S_n(1 - \beta) &= 1 - \beta^{n+1} \\
S_n &= \frac{1 - \beta^{n+1}}{1 - \beta},
\end{align*}
\]

so,

\[
\sum_{i=0}^{\infty} \beta^i = \lim S_n
\]

\[
= \lim \frac{1 - \beta^{n+1}}{1 - \beta}
\]

\[
= \frac{1}{1 - \beta} \text{ if } |\beta| < 1.
\]
Cauchy sequences

Definition
A sequence \( \{x_n\}_{n=1}^{\infty} \) is a **Cauchy** sequence if for any \( \epsilon > 0 \) \( \exists N \) such that for all \( i, j \geq N \), \( d(x_i, x_j) < \epsilon \).

Theorem
A sequence in \( \mathbb{R}^n \) converges if and only if it is a Cauchy sequence.

- Implied by least upper bound property of \( \mathbb{R} \)
- Proof in 29.1 of Simon and Blume
- Not always true, e.g. \( \mathbb{Q} \)
Completeness

Definition
A metric space, $X$, is complete if every Cauchy sequence of points in $X$ converges in $X$.

- **Banach space** = complete normed vector space.
- **Hilbert space** = complete inner product space.
Section 2

Open sets
Open sets

**Definition**
Let \( X \) be a metric space and \( x \in X \). A **neighborhood** of \( x \) is the set
\[
N_\epsilon(x) = \{ y \in X : d(x, y) < \epsilon \}.
\]

**Definition**
A set, \( S \subseteq X \) is **open** if \( \forall x \in S \), \( \exists \epsilon > 0 \) such that
\[
N_\epsilon(x) \subseteq S.
\]

- At any point in an open, can move slightly and stay in the set.
Open sets

Example

Open sets:

- \((a, b) = \{x \in \mathbb{R} : a < x < b\}\)
- \((\infty, b) = \{x \in \mathbb{R} : x < b\}\)
- The whole space
- \(\emptyset\)
- Open unit ball \(\{x \in \mathbb{R}^n : \|x\| < 1\}\)
Not open sets

Example

Not open sets:

- \([a, b) = \{ x \in \mathbb{R} : a \leq x < b \}\]
- \([a, b] = \{ x \in \mathbb{R} : a \leq x \leq b \}\]
- Any linear subspace of dimension \( k < n \) in \( \mathbb{R}^n \).
- Finite sets in \( \mathbb{R}^n \)
- \( \mathbb{Q} \subset \mathbb{R} \)
Open sets

Theorem

1. Any union of open sets is open. (finite or infinite)
2. The finite intersection of open sets is open.

Proof.
Let $S_j, j \in J$ be a collection of open sets. Pick any $j_0 \in J$. If $x \in \bigcup_{j \in J} S_j$, then there must be $\epsilon_{j_0} > 0$ such that $N_{\epsilon_{j_0}}(x) \subset S_{j_0}$. It is immediate that $N_{\epsilon_{j_0}}(x) \subset \bigcup_{j \in J} S_j$ as well.

Let $S_1, \ldots, S_k$ be a finite collection of open sets. For each $i$, $\exists \epsilon_i > 0$ such that $N_{\epsilon_i}(x) \subset S_i$. Let $\xi = \min_{i \in \{1, \ldots, k\}} \epsilon_i$. Then $\xi > 0$ since it is the minimum of a finite set of positive numbers. Also, $N_\xi(x) \subset S_i$ for each $i$, so $N_\xi(x) \subset \bigcap_{i=1}^k S_i$. \qed
Definition
The **interior** of a set $A$ is the union of all open sets contained in $A$. It is denoted as $\text{int}(A)$.

- The interior of a set is open
- Open sets are equal to their interior

Example
Interiors of sets in $\mathbb{R}$.

1. $A = (a, b)$, $\text{int}(A) = (a, b)$.
2. $A = [a, b]$, $\text{int}(A) = (a, b)$.
3. $A = \{1, 2, 3, 4, \ldots\}$, $A = \emptyset$
Section 3

Closed sets
Closed sets

**Definition**
A set $S \subseteq X$ is closed if its complement, $X^c$, is open.

**Example**
Closed sets:

1. $[a, b] \subseteq \mathbb{R}$
2. Any linear subspace of $\mathbb{R}^n$
3. $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$

Not closed sets:

1. Any open set
2. $\mathbb{Q} \subseteq \mathbb{R}$
Closed sets

Theorem

1. The intersection of any collection of closed sets is closed.
2. The union of any finite collection of closed sets is closed.

Proof.
Let \( C_j, j \in J \) be a collection of closed sets. Then
\[
(\bigcap_{j \in J} C_j)^c = \bigcup_{j \in J} C_j^c.
\]
\( C_j^c \) are open, so by theorem 20,
\[
\bigcup_{j \in J} C_j^c = (\bigcap_{j \in J} C_j)^c = \text{is open}.
\]
The proof of part 2 is similar.
Closed sets

Theorem
Let \( \{x_n\} \) be any convergent sequence with each element contained in a set \( C \). Then \( \lim x_n = x \in C \) for all such \( \{x_n\} \) if and only if \( C \) is closed.

- This is usually a more useful definition of closed sets
Proof.

First, we will show that any set that contains the limit points of all its sequences is closed. Let \( x \in C^c \). Consider \( N_{1/n}(x) \). If for any \( n \), \( N_{1/n}(x) \subset C^c \), then \( C^c \) could be open. If for all \( n \), \( N_{1/n}(x) \not\subset C^c \), then \( \exists y_n \in N_{1/n}(x) \cap C \). The sequence \( \{y_n\} \) is in \( C \) and \( y_n \to x \). However, by assumption \( C \) contains the limit of any sequence within it. Therefore, there can be no such \( x \), and \( C^c \) must be open and \( C \) is closed.

Suppose \( C \) is closed. Then \( C^c \) is open. Let \( \{x_n\} \) be in \( C \) and \( x_n \to x \). Then \( d(x_n, x) \to 0 \), and for any \( \epsilon > 0 \), \( \exists x_n \in N_{\epsilon}(x) \).

Hence, there can be no \( \epsilon \) neighborhood of \( x \) contained in \( C^c \). \( C^c \) is open by assumption, so \( x \not\in C^c \) and it must be that \( x \in C \).
Closure

Definition
The **closure** of a set $S$, denoted by $\overline{S}$ (or $\text{cl}(S)$), is the intersection of all closed sets containing $S$.

- The closure of a set is closed
- A closed set is its own closure
- Examples
  - $(a, b) = [a, b]$
  - $\emptyset = \emptyset$
Lemma

\( \overline{S} \) is the set of limits of convergent sequences in \( S \).

Proof.

Let \( \{x_n\} \) be a convergent sequence in \( S \) with limit \( x \). If \( C \) is any closed set containing \( S \), then \( \{x_n\} \) is in \( C \) and by theorem 26, \( x \in C \). Therefore, \( x \in S \).

Let \( x \in \overline{S} \). For any \( \epsilon > 0 \), \( N_\epsilon(x) \cap S \neq \emptyset \) because otherwise \( N_\epsilon(x)^c \) is a closed set containing \( S \), but not \( x \). Therefore, we can construct a sequence \( x_n \in S \cap N_{1/n}(x) \) that converges to \( x \) and is in \( S \).
Boundary

Definition
The **boundary** of a set $S$ is $\overline{S} \cap \overline{S^c}$.

- Equivalently, $\overline{S} \setminus \text{int}(S)$
- Boundary can be empty, e.g. $\mathbb{Q} \subset \mathbb{R}$, $\emptyset$
Lemma

If $x$ is in the boundary of $S$ then $\forall \epsilon > 0$, $N_\epsilon(x) \cap S \neq \emptyset$ and $N_\epsilon(x) \cap S^c \neq \emptyset$.

Proof.

As in the proof of lemma 28, all $\epsilon$-neighborhoods of $x \in \overline{S}$ must intersect with $S$. The same applies to $S^c$. 

\qed
Section 4

Compact sets
Open cover

Definition

An open cover of a set $S$ is a collection of open sets, $\{G_\alpha\}$ $\alpha \in \mathcal{A}$ such that $S \subseteq \bigcup_{\alpha \in \mathcal{A}} G_\alpha$.

Example

Some open covers of $\mathbb{R}$ are:

- $\{\mathbb{R}\}$
- $\{(-\infty, 1), (-1, \infty)\}$
- $\{\ldots, (-3, -1), (-2, 0), (-1, 1), (0, 2), (1, 3), \ldots\}$
- $\{(x, y) : x < y\}$

Example

Let $X$ be a metric space and $A \subseteq X$ and $\epsilon > 0$. $\{N_\epsilon(x)\}_{x \in A}$ is an open cover of $A$.
Compact sets

Definition
A set $K$ is **compact** if every open cover of $K$ has a finite subcover.

- Finite subcover means finite $G_{\alpha_1}, \ldots G_{\alpha_n}$ such that $K \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$
- e.g. $G_{\alpha} = (\alpha - \epsilon, \alpha + \epsilon)$ for $\alpha \in [0, 1]$ is an open cover of $[0, 1]$. One finite subcover is $(-\epsilon, \epsilon), (\epsilon/2 - \epsilon, \epsilon/2 + \epsilon), \ldots (1 - \epsilon, \epsilon)$ i.e. $G_{\alpha_i} = (i\epsilon/2 - \epsilon, i\epsilon/2 + \epsilon)$
Example

\( \mathbb{R} \) is not compact.

\( \{\ldots, (-3, -1), (-2, 0), (-1, 1), (0, 2), (1, 3), \ldots\} \) is an infinite cover, but if we leave out any single interval (the one beginning with \( n \)) we will fail to cover some number \( (n + 1) \).
Example

Let $K = \{x\}$, a set of a single point. Then $K$ is compact. Let $\{G_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of $K$. Then $\exists \alpha$ such that $x \in G_\alpha$. This single set is a finite subcover.
Example

Let $K = \{x_1, \ldots, x_n\}$ be a finite set. Then $K$ is compact. Let $\{G_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of $K$. Then for each $i$, $\exists \alpha_i$ such that $x_i \in G_{\alpha_i}$. The collection $\{G_{\alpha_1}, \ldots G_{\alpha_n}\}$ is a finite subcover.
Example

$(0, 1) \subseteq \mathbb{R}$ is not compact. $\{(1/n, 1)\}_{n=2}^{\infty}$ is an open cover, but there can be no finite subcover. Any finite subcover would have a largest $n$ and could not contain, e.g. $1/(n + 1)$. 
Compact sets

- Definition a bit abstract, but will show that a set in $\mathbb{R}^n$ is compact iff it is closed and bounded, which we will prove in the next few slides

**Definition**

Let $X$ be a metric space and $S \subseteq X$. $S$ is **bounded** if

$\exists x_0 \in S$ and $r \in \mathbb{R}$ such that

$$d(x, x_0) < r$$

for all $x \in S$. 
Lemma
Let $X$ be a metric space and $K \subseteq X$. If $K$ is compact, then $K$ is closed.

Proof.
Let $x \in K^c$. The collection $\{N_d(x,y)/3(y)\}$, $y \in K$ is an open cover of $K$. $K$ is compact, so there is a finite subcover, $N_d(x,y_1)/3(y_1), ..., N_d(x,y_n)/3(y_n)$. For each $i$, $N_d(x,y_i)/3(y_i) \cap N_d(x,y_i)/3(x) = \emptyset$, so

$$\cap_{i=1}^{n} N_d(x,y_i)/3(x)$$

is an open neighborhood of $x$ that is contained in $K^c$. $K^c$ is open, so $K$ is closed.
Lemma

Let $K \subseteq X$ be compact. Then $K$ is bounded.

Proof.

Pick $x_0 \in K$. \{\(N_r(x_0)\)\}_{r \in \mathbb{R}} is an open cover of $K$, so there must be a finite subcover. The finite subcover has some maximum $r^\ast$. Then $K \subseteq N_{r^\ast}(x_0)$, so $K$ is bounded.
Theorem (Heine-Borel)

A set $S \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof.

1. Compact $\Rightarrow$ closed and bounded shown by last two lemmas (regardless of whether $S \subseteq \mathbb{R}^n$ or some other space)
Proof.

2 Closed and bounded \( \Rightarrow \) compact

- Bounded so subset of cube, \([−a, a]^n\), for some \(a\)

2.1 Show \([-a, a]^n\) compact

- Suppose not, infinite cover of \([-a, 0]\) or \([0, a]\), repeat \(k\) times to get infinite cover of closed interval \(I_k\) of length \(a2^{-k}\)
- \(\bigcap_{k=1}^{\infty} I_k \neq \emptyset\) because \(I_k\) closed and nested
- But eventually \(I_k \subset N_\epsilon(x)\) for any \(\epsilon > 0\) and we have a finite subcover

2.2 Closed subset of compact set is compact
- Always compact $\Rightarrow$ close and bounded
- In $\mathbb{R}^n$ closed and bounded $\Rightarrow$ compact
- In infinite dimensional spaces, a set can be closed and bounded but not compact

Example

$\ell^\infty = \{(x_1, x_2, \ldots) : \sup_i |x_i| < \infty \text{ with norm } \|x\| = \sup_i |x_i|$

- $e_i = \text{all 0s except for a 1 in the } i\text{th position}$
- $E = \{e_i\}_{i=1}^\infty$ is closed and bounded
- $E$ is not compact
Sequential compactness

**Definition**
Let $X$ be a metric space and $K \subseteq X$. $K$ is **sequentially compact** if every sequence in $K$ has an accumulation point in $K$.

**Example**
- $[0, 1] \subseteq \mathbb{R}$ is sequentially compact
- $\mathbb{N} \subseteq \mathbb{R}$ is not sequentially compact
- $(0, 1) \subseteq \mathbb{R}$ is not sequentially compact
Compact $\Rightarrow$ sequentially compact

Lemma

Let $X$ be a metric space and $K \subseteq X$ be compact. Then $K$ is sequentially compact.

Proof.

- Given $\{x_n\}_{n=1}^{\infty}$, construct Cauchy sub-sequence:
  - Pick any $\epsilon > 0$, $N_\epsilon(x)$, $x \in K$ is an open cover of $K$, so there is a finite subcover, so $\exists x^* \text{ s.t. } \text{infinite } x_n \in N_\epsilon(x^*)$
  - Let $n_1$ be smallest $n$ s.t. $x_n \in N_\epsilon(x^*)$
  - Repeat with $\tilde{K} = \overline{N_\epsilon(x^*)} \cap K$ instead of $K$ and $\epsilon/2$ instead of $\epsilon$

- Conclude $K$ sequentially compact
Theorem

Let $X$ be a metric space and $K \subseteq X$. $K$ is compact if and only if $K$ is sequentially compact.

Proof.

- Already showed compact $\Rightarrow$ sequentially compact
- See notes for proof that sequentially compact $\Rightarrow$ compact
Theorem (Bolzano-Weierstrass)

A set $S \subseteq \mathbb{R}^n$ is closed and bounded if and only if it is sequentially compact.

- In $\mathbb{R}^n$, compact, sequentially compact, and closed and bounded are all equivalent

Corollary

Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.
Compactness

- $S \subseteq \mathbb{R}^n$ compact if
  1. Every open cover has a finite subcover
  2. Closed and bounded
  3. Every sequence in $S$ has a convergent subsequence with its limit in $S$

- In metric spaces that are not $\mathbb{R}^n$, 1 and 3 are still equivalent, but 2 is not