Single Agent Dynamic Models

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Economics 565

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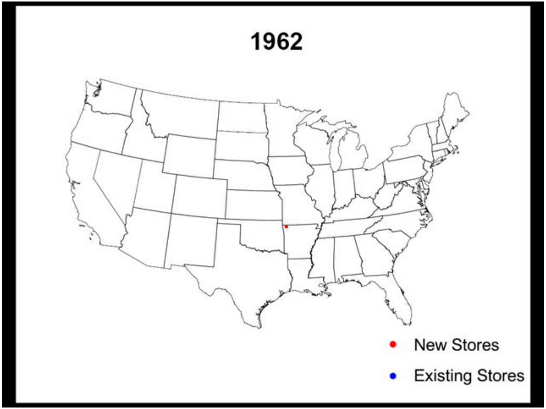
References

- Reviews:
  - Aguirregabiria (2021) chapters 6-7
  - Rust (2008)
  - Aguirregabiria and Mira (2010)
  - My notes from 628
- Key papers:
  - Rust (1987), Hotz and Miller (1993)

Section 1

Holmes (2011)

Holmes (2011): "The diffusion of Wal-Mart and economies of density"



Movie

Spread of Walmarts

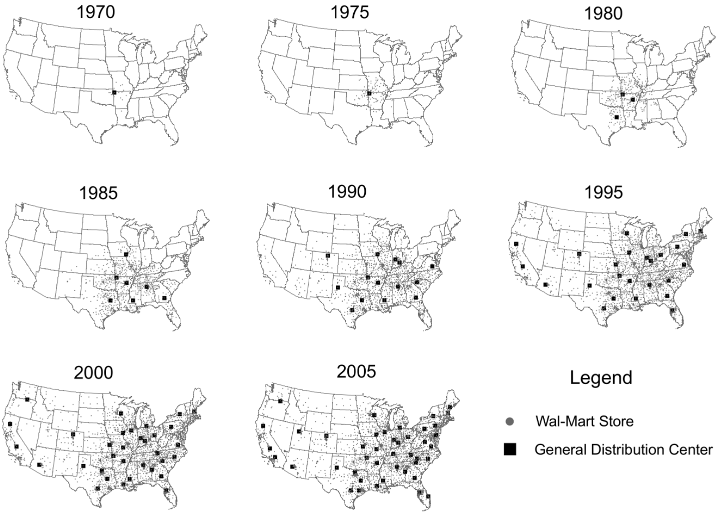


FIGURE 1.—Diffusion of Wal-Mart stores and general distribution centers.

Spread of Walmart Supercenters

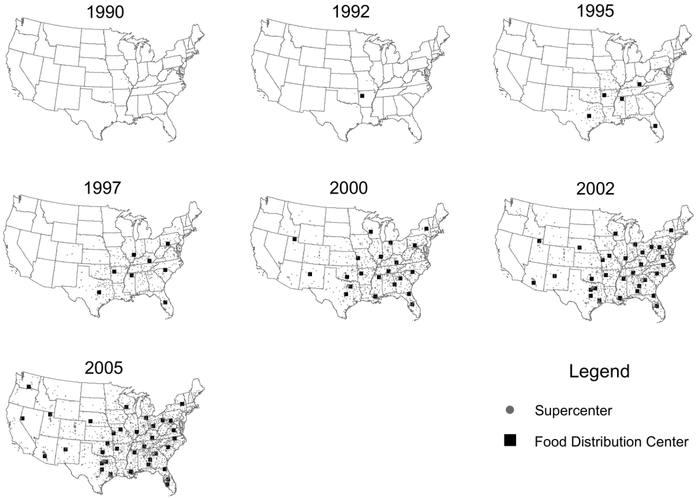


FIGURE 2.—Diffusion of supercenters and food distribution centers.

Spread of Walmarts

TABLE II

DISTRIBUTION OF WAL-MART FACILITY OPENINGS BY DECADE AND OPENING TYPE[a]

	General Merchandise (Including Supercenters)		Food Store (Part of Supercenter)		General Distribution Centers		Food Distribution Centers	
Decade Open	Opened This Decade	Cumulative	Opened This Decade	Cumulative	Opened This Decade	Cumulative	Opened This Decade	Cumulative
1962–1969	15	15	0	0	1	1	0	0
1970–1979	243	258	0	0	1	2	0	0
1980–1989	1,082	1,340	4	4	8	10	0	0
1990–1999	1,130	2,470	679	683	18	28	9	9
2000–2005	706	3,176	1,297	1,980	15	43	26	35

[a]Source: See Appendix A.

Holmes (2011) overview 1

- Observation: Walmart[1] opens its new stores close to existing ones
- Benefit from high store density: distribution
  - Shipping costs
  - Rapid response to demand shocks
- Question: how large are the benefits of density for Walmart?
- Challenge: Wal-Mart logistics data is confidential, even if detailed cost data available some benefits of density might not be reflected by it
- Solution: use revealed preference
  - Walmart's choices reveal tradeoff between benefit and cost of density

[1] Should it be "Wal-Mart" or "Walmart"?

Holmes (2011) 1

- Cost of high store density: cannibalization
  - Two Walmarts close together will take sales away from one another
  - Can be inferred from demand estimates
- Sequence of store openings important, so need a dynamic model
- Walmart's dynamic decisions:
  1. How many new Walmarts and how many new supercenters should be opened?
  2. Where should the new Walmarts and supercenters be located?
  3. How many new distribution centers should be opened?
  4. Where should the new distribution centers be located?

  Focus on 2 and take 1, 3, and 4 as given

Model: dynamic choice of locations 1

- Complete information
- Take as given choice of number of stores, N_t^{Wal}, and supercenters, N_t^{Super}
- Choose new store locations to maximize discounted sum of profits

$$\max_a \sum_{t=1}^{\infty} (\rho_t \beta)^{t-1} \begin{bmatrix} \sum_{j \in \mathcal{B}_t^{\text{Wal}}} (\pi_{jt}^g - c_{jt}^g - \tau d_{jt}^g) + \\ + \sum_{j \in \mathcal{B}_t^{\text{Super}}} (\pi_{jt}^f - c_{jt}^f - \tau d_{jt}^f) \end{bmatrix}$$

  - g, f superscripts for goods and food
  - π is variable profits

$$\pi_{jt}^e = \mu \underbrace{R_{jt}^e}_{\text{revenue}} - \text{Wage}_{jt} \text{Labor}_{jt}^e - \text{Rent}_{jt} \text{Land}_{jt}^e - \text{Other}_{jt}^e$$

Model: dynamic choice of locations 2

- R^e_{jt} = revenue comes from demand estimates; demand at store j depends on whether there is a store at nearby location k and through a distance term in consumers' utility of shopping at a store
- c_{jt} is a fixed cost

$$c_{jt} = \omega_0 + \omega_1 \log(\text{Popden}_{jt}) + \omega_2 \log(\text{Popden}_{jt})^2$$

- d_{jt} is distance to the nearest distribution center, τd_{jt} is a (fixed) distribution cost
- $\mathcal{B}^{\text{Wal}}_t$ is set of all Walmarts open at time t
- $\mathcal{B}^{\text{Super}}_t \subset \mathcal{B}^{\text{Wal}}_t$ is set of Walmart Supercenters
- $a = (\mathcal{A}^{\text{Wal}}_1, \mathcal{A}^{\text{Super}}_1, \ldots)$ is sequence of sets of new stores
- Stores never close

$$\begin{aligned} \mathcal{B}^{\text{Wal}}_t &= \mathcal{B}^{\text{Wal}}_{t-1} + \mathcal{A}^{\text{Wal}}_t \\ \mathcal{B}^{\text{Super}}_t &= \mathcal{B}^{\text{Super}}_{t-1} + \mathcal{A}^{\text{Super}}_t \end{aligned}$$

Model: demand

- Consumers make discrete choice among Walmarts within 25 miles and an outside option

$$u_{i0} = \alpha_0 + \alpha_1 \log(\text{Popden}_{l(i)}) + \alpha_2 \log(\text{Popden}_{l(i)})^2 + \epsilon_{i0}$$
$$u_{ij} = (\xi_0 + \xi_1 \log(\text{Popden}_{l(i)}))\text{Distance}_{l(i)j} + \text{StoreChar}_j \gamma + \epsilon_{ij}$$

  - $l(i)$ = location of consumer i
  - $\epsilon_i \sim$ logit
- Revenue:

$$R_j^g = \sum_l \underbrace{\lambda^g}_{\text{spending per } i} \times \underbrace{p_{jl}^g}_{\mathbb{P}(l \text{ shops at } j)} \times \underbrace{n_l}_{\text{number of consumers}}$$

- Revenue data is store-level sales estimate from Trade Dimensions, so must have measurment error

$$\log(R_j^{\text{Data}}) = \log(R_j^{\text{True}}) + \eta_j^{\text{Sales}}$$

where $\eta_j^{\text{Sales}} \sim N$

Estimation strategy

1. Estimate revenue using demand model and data on store sales
2. Construct variable costs based on local wages and property values
3. Estimate fixed costs (ω) and densities of scale (τ) using moment inequalities derived from profit maximization

Data

- Store level sales and employment in 2005
- Store openings and locations
- Demographic data from census
- Local wages and land rents
- Information from Walmart's annual reports

TABLE IV
PARAMETER ESTIMATES FOR DEMAND MODEL

Parameter	Definition	Unconstrained	Constrained (Fits Reported Cannibalizatio
λ^g	General merchandise spending per person (annual in \$1,000)	1.686 (.056)	1.938 (.043)
λ^f	Food spending per person (annual in \$1,000)	1.649 (.061)	1.912 (.050)
ξ_0	Distance disutility (constant term)	.642 (.036)	.703 (.039)
ξ_1	Distance disutility (coefficient on $\ln(\underline{\text{Popden}})$)	−.046 (.007)	−.056 (.008)
α	Outside alternative valuation parameters		
	Constant	−8.271 (.508)	−7.834 (.530)
	$\ln(\underline{\text{Popden}})$	1.968 (.138)	1.861 (.144)
	$\ln(\underline{\text{Popden}})^2$	−.070 (.012)	−.059 (.013)
	Per capita income	.015 (.003)	.013 (.003)
	Share of block group black	.341 (.082)	.297 (.076)
	Share of block group young	1.105 (.464)	1.132 (.440)
	Share of block group old	.563 (.380)	.465 (.359)
γ	Store-specific parameters		
	Store age 2 + dummy	.183 (.024)	.207 (.023)
σ^2	Measurement error	.065 (.002)	.065 (.002)
N		3,176	3,176
Sum of squared error		205.117	206.845
R^2		.755	.753
(Likelihood)		−155.749	−169.072

TABLE V

CANNIBALIZATION RATES, FROM ANNUAL REPORTS AND IN MODEL[a]

Year	From Annual Reports	Demand Model (Unconstrained)	Demand Model (Constrained)
1998	n.a.	.62	.48
1999	n.a.	.87	.67
2000	n.a.	.55	.40
2001	1	.67	.53
2002	1	1.28	1.02
2003	1	1.38	1.10
2004	1	1.43	1.14
2005	1	1.27	1.00[b]

[a]Source: Estimates from the model and Wal-Mart Stores, Inc. (1971–2006) (Annual Reports 2004, 2006).

[b]Cannibalization rate is imposed to equal 1.00 in 2005.

TABLE VI

COMPARATIVE STATICS WITH DEMAND MODEL[a]

Distance (Miles)	Population Density (Thousands of People Within a 5-Mile Radius) 1	5	10	20	50	100	250
0	.999	.989	.966	.906	.717	.496	.236
1	.999	.979	.941	.849	.610	.387	.172
2	.997	.962	.899	.767	.490	.288	.123
3	.995	.933	.834	.659	.372	.206	.086
4	.989	.883	.739	.531	.268	.142	.060
5	.978	.803	.615	.398	.184	.096	.041
10	.570	.160	.083	.044	.020	.011	.006

[a] Uses constrained model.

Variable costs

- Labor costs = average employees per million dollars of sales (3.61) (measure in 2005) $\times$ average retail wage in county in year
- Land value to sales ratio constructed from property values based on census data (for each year) and property tax data for Walmarts in Minnesota and Iowa
- Scale demand estimates from 2005 by average Walmart revenue in each year

Dynamic estimation 1

- Given demand estimates and variable costs only unknown parameters are fixed costs, ω, and economies of density, τ
- Total profits from action a = variable profits plus fixed costs plus economics of density

$$\Pi^T(a) = \sum_{t=1}^{\infty} (\rho_t \beta)^{t-1} \left[\begin{array}{l} \sum_{j \in \mathcal{B}_t^{\text{Wal}}} (\pi_{jt}^g - (\omega_0 + \omega_1 \log(\text{Popden}_{jt}) + \omega_2 \log(\text{Popde} \\ + \sum_{j \in \mathcal{B}_t^{\text{Super}}} (\pi_{jt}^f - (\omega_0 + \omega_1 \log(\text{Popden}_{jt}) + \omega_2 \log(\text{Pop} \end{array} \right.$$

$$= \Pi(a) + \omega_0 + \omega_1 C_{1,a} + \omega_2 C_{2,a} + \tau D_a$$

- Profit maximization implies that

$$\Pi^T(a) \leq \Pi^T(a^o)$$

$$\underbrace{\omega_1(C_{1,a} - C_{1,a^o}) + \omega_2(C_{2,a} - C_{2,a^o}) + \tau(D_a - D_{a^o})}_{\equiv x_a' \theta} \leq \underbrace{\Pi(a^o) - \Pi(a)}_{\equiv y_a}$$

where a^o is observed choice, a is any other choice

Dynamic estimation 2

- Estimation of demand and variable costs $\Rightarrow$ observe y_a with error
- Assume measurement has zero mean given x_a, then conditional moment inequalities,

$$\mathrm{E}[y_a - x_a'\theta | x_a] \geq 0$$

  can be used to form objective function
- Must choose deviations a and unconditional moment inequalities for estimation
  - Uses pairwise resequencing deviations (i.e. change order a pair of stores opens)
  - Group deviations according to their affect on density to aggregate conditional moment inequalities

TABLE XI

BASELINE ESTIMATED BOUNDS ON DISTRIBUTION COST τ[a]

	Specification 1 Basic Moments (12 Inequalities)		Specification 2 Basic and Level 1 (84 Inequalities)		Specification 3 Basic and Levels 1, 2 (336 Inequalities)	
	Lower	Upper	Lower	Upper	Lower	Upper
Point estimate	3.33	4.92	3.41	4.35	3.50	3.67
Confidence thresholds						
With stage 1 error correction						
PPHI inner (95%)	2.69	6.37	2.89	5.40	3.01	4.72
PPHI outer (95%)	2.69	6.41	2.86	5.45	2.97	5.04
No stage 1 correction						
PPHI inner (95%)	2.84	5.74	2.94	5.11	3.00	4.62
PPHI outer (95%)	2.84	5.77	2.93	5.13	2.99	4.97

[a]Units are in thousands of 2005 dollars per mile year; number of deviations $M = 522,967$; number of store locations $N = 3,176$.

TABLE XII

MEAN INCREMENTAL MILES SAVED AND STORES SERVED FOR DISTRIBUTION CENTERS ACROSS ALTERNATIVE OPENING DATES INCLUDING ACTUAL

	1 Year Prior to Actual	Actual Year Opened	1 Year After Actual	2 Years After Actual
All distribution centers ($N = 78$)				
Mean incremental miles saved	4.4	5.8	6.7	7.1
Mean stores served	23.6	52.1	58.4	62.9
By type of DC				
Regional distribution centers ($N = 43$)				
Mean incremental miles saved	6.1	7.7	8.7	8.9
Mean stores served	37.1	68.6	76.1	79.0
Food distribution centers ($N = 35$)				
Mean incremental miles saved	2.3	3.4	4.3	5.0
Mean stores served	6.9	31.8	36.5	43.0

- $\tau \approx \$3.50 =$ cost savings per year in thousands of dollars when a store is 1 mile closer to its distribution center
    - Shipping costs $\approx \$0.85$
- Results robust to splitting sample, changing revenues, labor, or rent, including/excluding supercenters

Section 2

General Setup

General Setup

- Discrete time t, maximum $\tau \leq \infty$
- State $s_{it} \in S$, follows a controlled Markov process

$$F(s_{it+1}|\mathcal{I}_t) = F(s_{it+1}|s_{it}, a_{it})$$

- Action $a_{it} \in A$
- Preferences: $\sum_{j=0}^{\infty} \beta^j U(a_{i,t+j}, s_{i,t+j})$

$$a_{it} \in \arg\max_{a \in A} \mathrm{E}\left[\sum_{j=0}^{\tau} \beta^j U(a_{i,t+j}, s_{i,t+j}) | a_{it} = a, s_{it}\right].$$

- Bellman equation

$$V(s_{it}) = \max_{a \in A} U(a, s_{it}) + \beta \mathrm{E}[V(s_{i,t+1})|a, s_{it}]$$

- Policy function

$$\alpha(s) = \arg\max_{a \in A} U(a, s_{it}) + \beta \mathrm{E}[V(s_{i,t+1})|a, s_{it}]$$

Example: Retirement

Example (Retirement)

[2] Consider the choice of when to retire. Let $a_{it} = 1$ if an agent is working and $a_{it} = 0$ if retired. Suppose τ is the age at death. The payoff function could be

$$U(a_{it}, x_{it}, \epsilon_{it}) = \mathrm{E}[c_{it}^{\theta_1} | a_{it}, x_{it}] \exp\left(\theta_2 + \theta_3 h_{it} + \theta_4 \frac{t}{1+t}\right) - \theta_5 a_{it} + \epsilon($$

where c_{it} is consumption, θ_1 is the coefficient of relative risk aversion, h_{it} is health, and the expression in the $\exp$ captures the idea that the marginal utility of consumption could vary with health and age. $-\theta_5 a_{it}$ captures the disutility of working.

[2] From Aguirregabiria and Mira (2010).

Example: Entry / Exit

Example (Entry/Exit)

[3] A firm is deciding whether to operate in a market. Its per-period profits are

$$U(a_{it}) = a_{it} \left(\theta_R \log(S_t) - \theta_N \log\left(1 + n_t\right) - \theta_F - \theta_E (1 - a_{i,t-1}) + \epsilon_{it} \right)$$

where a_{it} is whether the firm operates at time t. S_t is the size of the market, n_t is the number of other firms operating. θ_F is a fixed operating cost, and θ_E is an entry cost.

[3] From Aguirregabiria and Mira (2010).

Identification: Setup

- Panel data on N individuals for T periods
- Observe
  - Actions a_{it}
  - Some state variables x_{it}, $s_{it} = (x_{it}, \epsilon_{it})$
- i.e. observe joint distribution of $x_{i\cdot}$ and $a_{i\cdot}$
- Goal: recover U, $F(s_{it+1}|s_{it}, a_{it})$, β

Non-identification without more restrictions 1

- Value function

$$V(s) = \max_{a \in A} U(a,s) + \beta \mathrm{E}[V(s'|a,s)]$$

- Change $U(a,s)$ to

$$\tilde{U}_f(a,s) = U(a,s) + f(s) - \beta \mathrm{E}[f(s')|a,s]$$

, new value function

$$\tilde{V}(s) = \max_{a \in A} U(a,s) + f(s) - \beta \mathrm{E}[f(s')|a,s] + \beta \mathrm{E}[\tilde{V}(s')|a,s]$$

$$\tilde{V}(s) - f(s) = \max_{a \in A} U(a,s) + \beta \mathrm{E}[\tilde{V}(s') - f(s')|a,s]$$

- So, $V(s) = \tilde{V}(s') - f(s')$

Non-identification without more restrictions 2

- Policy functions,

$$\alpha(s) = \arg\max_{a \in A} U(a,s) + \beta \mathrm{E}[V(s'|a,s)]$$
$$\tilde{\alpha}(s) = \arg\max_{a \in A} U(a,s) + \beta \mathrm{E}[\tilde{V}(s') - f(s')|a,s]$$

  so $\alpha(s) = \tilde{\alpha}(s)$

- U leads to same policy as $\tilde{U}_f$; they are observationally equivalent

Identification: discrete A

- Assume:
    - A is discrete and finite
    - $U(a, x, \epsilon) = u(a, x) + \epsilon(a)$,
    - ϵ has known CDF G, $\epsilon \perp\!\!\!\perp x$ and $\epsilon_{it} \perp\!\!\!\perp \epsilon_{is}$ for $t \neq s$
        - Partial identification if G unknown, see Norets and Tang (2013)
    - β is known
    - $u(0, x) = 0$
- Then $u(a, x)$ is identified

Identification – discrete controls

- Given $\{x_{it}, a_{it}\}$ want to recover U
- Compare with discrete control identification from Magnac and Thesmar (2002) or Bajari, Chernozhukov, Hong, and Nekipelov (2009)
  - Assume:
    1. Transition distribution is identified
    2. Payoff additively separable in ϵ, $U(x, a, \epsilon) = u(x, a) + \epsilon(i)$
    3. Distribution of ϵ known and ϵ_{it} independent across f and t
    4. $u(x, a_0)$ is known for all $x \in \mathcal{X}$ and some $a_0 \in \mathcal{A}$
    5. Discount factor, δ, is known

Identification – discrete controls

- Proof sketch:
  - Additive separability and knowing distribution of ϵ allows Hotz-Miller inversion to recover differences of choice specific value functions
  - Given $u(x, a_0)$ and differences in choice specific value functions, can recover choice specific value functions from Bellman equation
  - Given choice specific value functions, can get $u(x, a)$ from Bellman equation
  - See my notes from 628 and references therein for details

Identification – continuous controls

- Key assumptions:
  1. Transition density, $f_{x_{it+1}|x_{it},a_{it}}$, is identified
  2. Distribution of ϵ, F_ϵ, is normalized
     - Not a restriction because ϵ enters $U(x, a, \epsilon)$ without restriction

     ϵ_{it} is independent across f and t.
  3. Discount factor, δ, is known
  4. For some k,
     - $\frac{\partial U}{\partial x^{(k)}}(x, \alpha(x, \epsilon), \epsilon)$ is known
     - There exists $\chi_k(a_0, x^{(-k)}, \epsilon)$ such that $a_0 = \alpha\left(\chi_k(a_0, x^{(-k)}, \epsilon), x^{(-k)}, \epsilon\right)$
  5. Initial condition: for some a_0, $U(x, a_0, \epsilon)$ is known

Identification – continuous controls

- Key assumptions (continued):
  6. Completeness: let $\frac{\partial U}{\partial a} \in \mathcal{G}$, define

$$\mathcal{D}(g)(x,\epsilon) = \frac{\partial}{\partial a_t} E\left[\sum_{\tau=0}^{\infty} \delta^\tau g(x_{t+\tau}, \epsilon_{t+\tau}) | x_t = x, a_t = \alpha(x,\epsilon)\right]$$

$$\mathcal{L}(g)(x,\epsilon) = \int_{\chi_k(a_0, x^{(-k)},\epsilon)}^{x^{(k)}} g(\tilde{x}^{(k)}, x^{(-k)}, \epsilon) \frac{\partial \alpha}{\partial x^{(k)}} (\tilde{x}^{(k)}, x^{(-k)}, \epsilon) d\tilde{x}^{(k)}$$

$$\mathcal{K}(g)(x,\epsilon) = \mathcal{D}(\mathcal{L}(g))(x,\epsilon)$$

  The only solution in $\mathcal{G}$ to

$$0 = g(x,\epsilon) + \mathcal{K}(g)(x,\epsilon)$$

  is $g(x,\epsilon) = 0$
- Result: U identified

Proof sketch

- Policy function: $F_\epsilon(\epsilon) = F_{a|x}\left(\alpha(x, \epsilon) \middle| x\right)$
- First order condition for a_t:

$$
\begin{aligned}
0 = & \frac{\partial U}{\partial a}(x_t, \alpha(x_t, \epsilon_t), \epsilon_t) + \\
& + \frac{\partial}{\partial a} \sum_{\tau=1}^{\infty} \delta^\tau E\left[U\left(x_{t+\tau}, \alpha(x_{t+\tau}, \epsilon_{t+\tau}), \epsilon_{t+\tau}\right) \middle| x_t, \alpha(x_t, \epsilon_t)\right]
\end{aligned}
$$

- Write payoff function in terms of its derivatives:

$$
\begin{aligned}
U(x, \alpha(x, \epsilon), \epsilon) = & \int_{\chi_k(a_0, x^{(-k)}, \epsilon)}^{x^{(k)}} \left(\frac{\partial U}{\partial a}(x, \alpha(x, \epsilon), \epsilon) \frac{\partial \alpha}{\partial x^{(k)}}(x, \epsilon) + \right. \\
& \left. + \frac{\partial U}{\partial x^{(k)}}(x, \alpha(x, \epsilon), \epsilon) d\tilde{x}^{(k)} \right) \\
& + U(\chi_k(a_0, x^{-k}, \epsilon), x^{-k}, a_0, \epsilon)
\end{aligned}
$$

Proof sketch

- Let

$$\varphi(x,\epsilon) = U(\chi_k(a_0, x^{-k}, \epsilon), x^{-k}, a_0, \epsilon) + \\ + \int_{\chi_k(a_0, x^{(-k)}, \epsilon)}^{x^{(k)}} \frac{\partial U}{\partial x^{(k)}}(x, \alpha(x,\epsilon), \epsilon) d\tilde{x}^{(k)}$$

- Substitute into first order condition:

$$0 = (\mathbf{1} + \mathcal{K})(\frac{\partial U}{\partial a}) + \mathcal{D}(\varphi)$$

- Integrate to recover $U(x, \alpha(x,\epsilon), \epsilon)$

Section 3

Machine replacement models

Machine replacement models

- Firm operates many machines independently, machines fail with some probability that increases with age, firm chooses when to replace machines to minimize costs of failure and replacement
- Classic Rust (1987) about bus-engine replacement
- Many follow-ups and extensions
  - Das (1992): cement kilns
  - Kennet (1994): aircraft engines
  - Rust and Rothwell (1995): nuclear power plants
  - Adda and Cooper (2000): cars
  - Kasahara (2009): response of investment to tariffs

Model

- Choose: $a_{it} = 1$ (replace) or 0 (don't replace)
- Machine age: $x_{it+1} = (1 - a_{it})x_{it} + \xi_{i,t+1}$
- Profits: $Y(x) - aRC(x) + \epsilon(a)$
- Firm's problem:

$$\max_{\mathbf{a}} \mathrm{E}_t \left[\sum_{j=0}^{\infty} \beta^j \left(Y\left((1 - a_{it})x_{it}\right) - a_{it}RC(x_{it}) + \epsilon_{it}(a_{it}) \right) \right]$$

$$\text{s.t. } x_{i,t+1} = (1 - a_{it})x_{it} + \xi_{i,t+1}$$

- ϵ and ξ i.i.d.
- Often ξ non-stochastic, e.g. $x =$ age, $\xi = 1$.

Value functions

- Value function

$$V(x,\epsilon) = \max_a Y((1-a)x) - aRC(x) + \epsilon(a) + \beta \mathrm{E}[V(x',\epsilon')|x(1-a)]$$

- Expected (or integrated) value function

$$\bar{V}(x) = \mathrm{E}\left[V(x',\epsilon')|x\right]$$

- Choice specific value function

$$\begin{aligned} v(x,a) = & Y((1-a)x) - aRC(x) + \beta \mathrm{E}\left[\max_{a'} v(x',a') + \epsilon(a')|x,a\right] \\ = & Y((1-a)x) - aRC(x) + \beta \bar{V}(x(1-a)) \end{aligned}$$

Identification 1

- Observe: $\mathrm{P}(a|x)$

$$
\begin{aligned}
\mathrm{P}(a=1|x) =& \mathrm{P}\left(v(x,1)+\epsilon(1) \geq v(x,0)+\epsilon(0)|x\right) \\
=& \mathrm{P}\left(\epsilon(0)-\epsilon(1) \leq v(x,1)-v(x,0)\right) \\
=& \mathrm{P}\left(\epsilon(0)-\epsilon(1) \leq Y(0)-Y(x)-RC(x)+\beta\left(\bar{V}(0)-\bar{V}(x)\right)\right)
\end{aligned}
$$

- Choice probabilities identify $v(x,1)-v(x,0)=\log \mathrm{P}(1|x)-\log \mathrm{P}(0|x)$
- Choice probabilities not enough to separately identify $RC(x)$ and $Y(x)$, only identify the sum $RC(x)+Y(x)$
- Normalize $Y(x)$, solve for $v(x,0)$ from

$$
\begin{aligned}
v(x,0) =& Y(x)+\beta \mathrm{E}\left[\max_a v(x',a)-v(x',0)+\epsilon(a)|x\right]+\beta \mathrm{E}[v(x',0)|x] \\
v(x,0) =& (I-\beta\mathcal{E})^{-1}\left(Y(\cdot)+\beta\mathrm{E}\left[\max_a v(x',a)-v(x',0)+\epsilon(a)|\cdot\right]\right)(x) \\
=& (I-\beta\mathcal{E})^{-1}\left(Y(\cdot)+\beta\mathrm{E}\left[\log\left(\sum_a e^{v(x',a)-v(x',0)}\right)|\cdot\right]\right)(x)
\end{aligned}
$$

where $\mathcal{E}(f)(x)=\mathrm{E}[f(x')|x]$

Identification 2

- Then

$$v(x,1) = v(x,0) + [v(x,1) - v(x,0)]$$

and

$$\begin{aligned}\bar{V}(x) =& \mathrm{E}\left[\max_{a'} v(x',a') + \epsilon(a')|x\right] \\ =& \mathrm{E}\left[v(x',0) + \max_{a'} v(x',a') - v(x',0) + \epsilon(a')|x\right] \\ =& \mathrm{E}\left[v(x',0) + \log\left(\sum_a e^{v(x',a')-v(x',0)}\right)|x\right]\end{aligned}$$

and

$$RC(x) = -v(x,1) + \beta \bar{V}(0)$$

(Non)-identification of discount factor 1

- Given β and $Y(x)$, above steps identify $RC(x)$, change β and get a new observationally equivalent $RC(x)$
- Does it matter? Consider counterfactuals that change Y or distribution of ξ. Check whether $\frac{\partial P}{\partial Y}$, $\frac{\partial v}{\partial Y}$ depend on β.
- Identify β by having some components of x affect $\mathrm{E}[\cdot|x,a]$, but not Y or RC
    - Previous slide gives RC as a function of β, identify β from restriction on RC

Other models of dynamic demand

- Storable goods
  - Hendel and Nevo (2006)
  - Erdem, Imai, and Keane (2003)
- Durable goods
  - Gowrisankaran and Rysman (2009)
- Health care
  - Gilleskie (1998)

Section 4

Euler equations

Euler Equations for the Estimation of Dynamic Discrete Choice Structural Models

- Euler equations provide easier way to estimate dynamic continuous choice models than solving for value function
- Derive Euler equations for discrete choice model
    - Write problem in terms of choice probability instead of policy to make differentiable
- Reduces computation, but loses asymptotic efficiency

Continuous choice 1

- Exogenous state z with density $f(z_{t+1}|z_t)$
- Endogenous state $y_{t+1} = Y(a_t, y_t, z_t, z_{t+1})$
- Action a
- Bellman equation

$$V(y,z) = \max_a \pi(a,y,z) + \beta \int V(Y(a,y,z,z'),z')dF(z'|z)$$

- FOC

$$0 = \frac{\partial \pi}{\partial a} + \beta \int \frac{\partial V}{\partial y}\frac{\partial Y}{\partial a} dF(z'|z)$$

- Envelope theorem

$$\frac{\partial V}{\partial y} = \frac{\partial \pi}{\partial y} + \beta \int \frac{\partial V}{\partial y}\frac{\partial Y}{\partial y} dF(z'|z)$$

- Present value approach (solving for V)

Continuous choice 2

- $$\frac{\partial V}{\partial y} = \frac{\partial \pi}{\partial y} + \beta \mathcal{E}(\frac{\partial V}{\partial y})$$
  $$\frac{\partial V}{\partial y} = (I - \beta\mathcal{E})^{-1} \frac{\partial \pi}{\partial y}$$
- substitute into FOC
  $$0 = \frac{\partial \pi}{\partial a} + \beta \int (I - \beta\mathcal{E})^{-1} \frac{\partial \pi}{\partial y} \frac{\partial Y}{\partial a} dF(z'|z)$$
  and use to estimate derivatives of π
- Downsides:
  - Computational curse of dimensionality $(I - \beta\mathcal{E})$ is like $|\mathcal{Z}| \times |\mathcal{Z}|$ matrix, so costly to invert
  - Statistical curse of dimensionality: need to estimate expectations conditional on z

- Euler equation approach:
  - Assume $\frac{\partial Y}{\partial y} = H(a, y, z) \frac{\partial Y}{\partial a}$

Continuous choice 3

- Combine with envelope theorem and FOC to get

$$0 = \frac{\partial \pi}{\partial a_t} + \beta \int \left(\frac{\partial \pi}{\partial y_{t+1}} - H(a_{t+1}, y_{t+1}, z_{t+1}) \frac{\partial \pi}{\partial a_{t+1}} \right) \frac{\partial Y}{\partial a_t} dF(z_{t+1} | z$$

- Equivalently, solve

$$\max_{a_t, a_{t+1}} \pi(a_t, y_t, z_t) + \beta \int \pi(a_{t+1}, Y(a_t, y_t, z_t, z_{t+1}), z_{t+1}) dF(z_{t+1} | z_t)$$
$$\text{s.t.} Y(a_{t+1}, Y(a_t, y_t, z_t, z_{t+1}), z_{t+1}, z_{t+2}) = y^*_{t+2}(y_t, z_t, z_{t+1}, z_{t+2})$$

Euler equation for discrete choice 1

- Rewrite problem of choosing $a(x)$ to choosing $P(x)$ using 1-1 mapping between probabilities and threshold decision rules

$$a(x) = \mathbf{1}(v(a,x) - v(j,x) \geq \epsilon(j) - \epsilon(a))$$

iff

$$P(a,x) = \tilde{G}(v)$$

- $$W(x) = \max_P \sum_a P(a,x)\left(\pi(a,x) + \mathrm{E}_P[\epsilon(a)|a] + \int W(x')dF(x'|x,\right.$$

Euler equation for discrete choice 2

- Constrained two-period problem to get Euler equation

$$\max_{P_t, P_{t+1}} \pi^e(x_t, P_t) + \beta \int \pi^e(x_{t+1}, P_{t+1}) dF^e(x_{t+1}|x_t, P)$$
$$\text{s.t.} F^e(x_{t+2}|x_t, P_t, P_{t+1}) = F^e(x_{t+2}|x_t, P_t^*, P_{t+1}^*)$$

Application: cow replacement

- Out of time, see paper

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