The Relativity of Simultaneity

Let me begin with a little homily.

In non-technical presentations of the theories of relativity (even the best of them) the austere pictures of world-lines of clocks are sometimes humanized or anthropomorphized by adding a human “observer” to the world line to go along with the clock. As an eminent philosopher once remarked, this serves to add artistic verisimilitude to an otherwise bald and unconvincing narrative.

The addition of observers should be harmless, but it may not be. Our friends Alice and Bob busily consult their clocks, note when light pulses leave or arrive, and then calculate various quantities, as shall we. But sometimes the verb ‘see’ (or its cognates or closely related expressions) is used to describe their clock readings and the results of their calculations. Thus an author might say that Alice sees the light pulse arrives at time $t = 12$ and that Alice sees Bob’s clock running slow. But the latter sort of statement needs to be treated cautiously. Alice may calculate that Bob’s clock is, somehow, running more slowly than hers, but what
she sees is when light pulses from Bob’s clock arrives at her clock.

As an example of how matters can be mixed together, consider the sentence we find on page 107 of *GRAB*: “By apparent we now mean apparent to that observer who happens to see the events as occurring simultaneously.” To “see” events “as occurring simultaneously” is, we are told in the text, to make an *inference* from the equality of two temporal intervals, \( t_1 \) and \( t_2 \). We shall look at simultaneity in a bit more detail below.

With this preliminary caution in mind, let us populate our clock world line with “observers” and note what they see or conclude in various situations. We might suppose, for instance, that our observers wish to think about space and time in terms of the concepts we used in Aristotelian spacetime. To what extent are they able to do so, using the interval?

We imagine the familiar situation. We have a worldline (with a clock) and on this worldline we fix a point \( p \). We send out a pulse of light at some point \( r \). The pulse reflects from (or at) \( q \) and returns to the clock worldline at point \( s \). Since the clock assigns times to all points on its worldline, it gives us the times that \( r, p, \) and \( s \) occurred, \( T(r) \),
T(p), and T(s), amongst many others. We call the elapsed time between p and s (that is, T(s) - T(p)) \( t_1 \) and the elapsed time between r and p (T(p) - T(r)) \( t_2 \). All this is familiar.¹

What could we say about *when* q occurs, in terms of the readings of our clock? We know the total time that the light pulse takes to travel to q and back--\( t_1 + t_2 \). Since the speed of light in space is fixed, the pulse must take as much time to reach q as it does traveling back from q. So the pulse must reach q exactly \( \frac{1}{2}(t_1 + t_2) \) seconds after leaving r.

We know that p occurs exactly \( t_2 \) seconds after r, according to our clock. So a reasonable way to think of the time difference between p and q is that it is \( \frac{1}{2}(t_1 + t_2) - t_2 \). If we indicate the time difference in our usual way, as \( \Delta t \), we can write:

\[
\Delta t = \frac{1}{2}(t_1 - t_2).
\]

In this formula \( \Delta t \) measures the difference between the (estimated) time at which q occurs after r and the (measured) time at which p occurs after r. So if \( \Delta t > 0 \), then q would be held to occur

¹ And let’s use the same units as Geroch, seconds and light-seconds.
after p; if $\Delta t < 0$, q would be held to occur before p.

This formula tells us, then, how much later than p the event q occurred. If $\Delta t = 0$--that is, if $t_1 = t_2$, then our observer would say that there is no difference in the time that p and q occurred--that is, that they occurred at the same time. This observer would not “see” that p and q were simultaneous but would, rather, conclude that they were, based on the clock’s measurement of $t_1$ and $t_2$.

Let’s stay with this case for a moment. It should be clear that many points can be found that are simultaneous with p, since all that’s required is that $t_1 - t_2$. The more time before p the pulse is sent out--that is, the larger $t_2$ is--the further from p the other simultaneous point is. And the pulse can be sent out in any direction from the clock’s worldline. So we can construct a little hyperplane of nearby events all simultaneous with p, a local counterpart in relativistic spacetime of the global hyperplanes of simultaneity in Aristotelian and Galilean spacetime. See figure 55 in *GRAB*. 
Let’s continue thinking about the situation we’re describing, with p on a clock worldline and q simultaneous with p according to a clock that goes through p. Let us ask how far away from p the point q is?

We have already convinced ourselves that it must take time $1/2 (t_1 + t_2)$ for our light pulse to reach q from the worldline containing p. Since light travels at speed $c$ in space, the distance of q from p must be

$$\Delta x = \frac{c}{2} (t_1 + t_2).$$

Then we use the formulas for $\Delta x$ and $\Delta t$ in a little algebraic manipulation.

From the formula immediately above, we can derive that

$$\frac{\Delta x}{c} = \frac{1}{2} (t_1 + t_2).$$

We know already that

$$\Delta t = \frac{1}{2} (t_1 - t_2).$$

If we add these two formulas, we get

$$\frac{\Delta x}{c} + \Delta t = t_1.$$
And if we subtract the second from the first, we get

\[ \Delta x/c - \Delta t = t_2. \]

The last two formulas tell us that:

\[ t_1 \cdot t_2 = \left( \frac{\Delta x}{c} \right)^2 - (\Delta t)^2. \]

Since we assume that the quantity on the left hand side, the interval, is an invariant quantity, then the expression on the right hand side must also be invariant. It is rather neat, and expressed in terms of distances and times (or coordinates, if you like to think of it that way).

One sees this way of representing the interval in slightly different guises. One might take advantage of the fact that (in appropriate units) the speed of light, \( c \), is equal to 1 and drop if from the formula.\(^2\) In that case one might write

\[ (\Delta x)^2 - (\Delta t)^2 = (\Delta x')^2 - (\Delta t')^2. \]

\(^2\) If one follows this convenient course, one might find oneself from time to time puzzling over an equation in which the units on the two sides do not match. If the units matter, then one has to find and insert the missing `c's.
The same relation can appear in its infinitesimal form

\[(dx)^2 - (dt)^2 = (dx')^2 - (dt')^2.\]

Of course the interval is really a four-dimensional invariant, so it may be written out explicitly as

\[(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 - (dt)^2.\]

Or, if one is focussing on temporal rather than spatial relations, one can multiply this expression by -1. The result is often written as

\[(d\tau)^2 = (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2.\]

The new symbol, \(\tau\), is the lower-case Greek letter \(tau\).

Let us return now to our consideration of local hyperplanes of simultaneity. We have seen how Alice might set up hers, for some event \(p\) on her world line. But let us suppose that Bob too encounters that event on his world line. If his world line differs from that of Alice, we shall see that he must disagree as to what events are simultaneous with \(p\). This result is what is shown in the next figure, where I let “observers” have
straight or inertial world-lines to reduce the amount of graphic artistry involved. Results are often most firmly proved by analytic means, and spacetime pictures can be misleading. But this figure is not misleading and makes its point vividly.
What we see is, first of all, Alice (and her clock) sending out a light pulse at r and receiving it back from q at s. Since $t_1$, the interval of time from p to s, is equal to $t_2$, the interval of time from r to p, Alice knows that $\Delta t = 0$ and concludes that q occurs at the same time as p.

An examination of the figure makes it clear that Bob must have a different view of the matter. He must send his light pulse out to q from point $r'$, so that it can travel along with Alice’s light pulse. He notes its return at $s'$.

As Bob measures time intervals his $t_2$ is longer than Alice’s, and his $t_1$ is shorter. Since from Bob’s perspective $t_1 \neq t_2$, he cannot conclude that q occurs at the same time as p. Given that $t_2 > t_1$, Bob will find that $\Delta t < 0$ and so conclude that q occurs earlier than p. (Were Bob moving in the -x direction instead of the x-direction, he would find that $\Delta t > 0$ and hold that q occurs later than p.)