A substantivalist about space (time) basically holds the view that space (and time) exist independently of the things “in” them. This is has been the root notion of substance since Aristotle, but it’s hard to make this idea precise in the current context.

In our formulations of spacetime theories it is presumably the manifold $M$ that is taken to be (or to represent) spacetime and the geometric objects its “contents”. So the root idea of substantivalism is that spacetime can exist independently of its contents, but in the context of GTR it is not clear how to make this idea precise.

Eventually we will generate a necessary condition for (modern, manifold) Substantivalism—that is, a condition that must itself be true else (modern, manifold) Substantivalism is false.

The classical anti-substantivalist argument is the dilemma that Leibniz aimed at Newton (or Clarke):

(a) Allow that there are distinct states of affairs which no possible observation could distinguish, or
(b) Give up substantivalism.
In the hey-day of verificationism, the dilemma was felt to be very acute, but verificationism looks to be a failed program.

A modern version of this argument poses a different dilemma. A spacetime substantivalist, it is argued, must either

(a) Accept an unpalatable form of indeterminism, 
or
(b) Give up substantivalism.

Setting Up The Argument

The Hole Argument hinges on what Earman and Norton call the “Gauge Theorem”. It is (practically) another way of expressing the idea of general covariance.

Gauge Theorem. If \(<M,O_1,...,O_n>\) is a model of a local spacetime theory and \(h\) is a diffeomorphism from \(M\) onto \(M\), then the carried along tuple \(<M, h*O_1,...,h*O_n>\) is also a model of the theory.

First, what does this mean?

A local **spacetime theory** is the kind of theory we have been studying.
A **diffeomorphism** is just a function, in this case a function from points of M to points of M'. The distinguishing features of a diffeomorphism are (i) that it has an inverse and (ii) that both it and its inverse are differentiable.

We will usually be considering functions that map M to M itself. Such functions are called **automorphisms**. In the context of the hole argument, a diffeomorphism from a manifold M onto itself is sometimes called a **Leibniz shift**.

Now here’s a slightly tricky part. Suppose that the diffeomorphism h takes (or maps) the point p to the point h(p). Let the “carried along” object O* have at h(p) the values that O had at p.

So the geometric objects are in effect shifted (by h and the “carry along” operation) from points p, q, r to different points of the spacetime, h(p), h(q), h(r). (“Active” interpretation of h)

**If** you want to look at this operation in terms of coordinate changes (“passively”), then what we are doing is changing the coordinates at p from those which it is assigned in the original coordinate system to (in a new coordinate system) the coordinates originally assigned to hp. (These two
ways of looking at it are equivalent, as we have claimed in class.)

In a generally covariant formulation of a theory, the coordinate transformations that are allowed are those that can be obtained from a slightly larger set of functions, functions that (i) have an inverse and (ii) are such that it and its inverse are continuous. These are the homeomorphisms. Every diffeomorphism is a homeomorphism, because every differentiable function is continuous (but not *vice versa*).

In a generally covariant formulation of a theory, every allowed coordinate transformation takes a model to a model. When viewed actively (instead of passively), this is practically the Gauge Theorem above.

**The Argument**

Norton points out that a spacetime substantivalist must **deny** the following assertion, called Leibniz equivalence:

Two intertransformable models of a spacetime theory, such as \(<M,g>\) and \(<M,g'>\) represent the same possible world.


“If everything in the world were reflected East to West (or, better, translated 3 feet East), retaining all the relations between bodies, would we have [the same] world? The substantivalist must answer [no] since all the bodies of the world are now in different spatial locations, even though the relations between them are unchanged.” (“What Price Spacetime Substantivalism, p. 521)

We are now able to put this same basic idea more generally and precisely. It looks very much as if a spacetime substantivalist must deny the following principle:

*Leibniz Equivalence*: Diffeomorphic models represent the same physical world or situation.

The identity map, I, is a diffeomorphism. [By definition, \( I(p) = p \).] Now choose some small neighbourhood of a point of the spacetime. This neighbourhood is the hole, \( H \). A hole diffeomorphism, \( h \), is a diffeomorphism that is \( I \) outside the hole but differs smoothly from \( I \) inside the hole.
Given that the metric $g$ is, like all the other geometric objects, shifted by a hole diffeomorphism, we can see why Norton says:

Given the fullest specification of the spacetime outside the hole, the theory will be unable to determine the trajectory along which a particle in free fall will traverse the hole, even though its trajectory before and after the hole is known exactly.

Put another way, the (modern manifold) substantivalist is committed to the independent existence of each given point $p$, $q$, $r$ inside the hole. One can view the hole diffeomorphism $h$ as a kind of permutation, a function that (in the hole, smoothly) replaces $p$ with $h(p)$, $q$ with $h(q)$, etc.

The carry along ensures that whatever value a geometric object like $g$ or $T$ had at $p$ and $q$, a “new” object, like $h^*g$ or $h^*T$ has that same value at $h(p)$ and $h(q)$, etc. Therefore the manifold substantivalist can not predict what happens at any chosen point $p$ or $q$ in the hole, given the plethora of diffeomorphic models, even given complete knowledge as to what happens at all the other points in the spacetime outside the hole.
Does the particle’s geodesic go through p or through h(p)? Since both \( <M,g,T> \) and \( <M,h^*g,h^*T> \) are models (by the Gauge Theorem), if h is a hole diffeomorphism, they agree everywhere outside the hole. Inside the hole they differ, but since both are models of the theory, one can’t say which point will lie on the particle’s path.

This sort of failure of the “initial value problem,” even if the spacetime contains Cauchy surfaces, really is unsettling. There is no difference whatsoever as to what is observed or even what is observable between two models connected by a hole diffeomorphism. The sole difference is what happens a (specified, individual) spacetime points.

And the unsettling conclusion seems to arise as a result of adopting manifold substantivalism and the peculiar individuation of spacetime points that seems to be a corollary of denying Leibniz equivalence.